Dimensional Analysis

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1 Introduction

To understand and describe a phenomenon in physics or mechanics, it is necessary to determine which effect and physical quantities are important in the phenomenon. Investigation of mechanical or physical phenomena are intended to lead to some definite law or equation relating physical quantities. Both theoretical and experimental approaches to a given problem may be used. Frequently the theoretical approach leads to some equation which is too difficult to solve by mathematical means. Other phenomena may be awkward to investigate experimentally. When observation and measurement are used to determine the unknown, special techniques must be employed to ensure that any experiment is a faithful reproduction of the true phenomenon. Dimensional analysis gives information about the general form of a relation between some unknown and other variables in a physical problem.

2 Dimensional quantities

A dimension is a measure of physical quantity(without numerical values), while ***unit*** is a way to assign a number to that ***dimension***. For example , length is a dimension that is measured in units such as microns (μm), centimeters (cm), meters(m), kilometers(km) etc. Also, mass is a dimension that is measured in units such as gram (g), kilogram (kg) and and time t is a dimension that is measured in seconds (s), hours (hr), years. In fluid dynamics it is usual to regard the three dimensions: **mass(M)**, **length (L**),and **time (T**) as fundamental dimension which can be used to express dimension of another mechanical quantities, like velocity $[V] = L/T$, acceleration $[a] = L/T^2$, force $[F] = ML/t^2$, pressure $[p] = [F]/[L^2] = M/LT^2$, density $[\rho] = M/L^2$, where square brackets indicate ”dimension of”. The dimension of the another quantities in fluid dynamics, e.g the coefficient of viscosity μ, must be *derived* from the definition:

$$\text{Shear stress} = \tau = \mu \frac{\partial v}{\partial y}$$

so that

$$\frac{M}{LT^2} = [\mu]\frac{1}{T}$$

and hence

$$[\mu] = \frac{M}{LT}$$

It follows that

$$[\nu] = \left[\frac{\mu}{\rho}\right] = \frac{L^2}{T}$$

Dimension of further quantities arising in fluid dynamics can be evaluated in a similar manner. It is not hard to note that dimension of all the quantities discussed so far are in the form of *monomial powers*

$$M^{\alpha_1}L^{\alpha_2}T^{\alpha_3}.$$

This property is true for all physical quantities.
One can regard that all physical quantities $(A, B, C, \ldots X)$ belong to the **dimensional space** Π. Following axioms for the *Pi* space are fulfilled:

1. $AB = BA$
2. $(AB)C = A(BC)$
3. the solution X of $AX = B$ exits for any pair A, B of elements of Π
4. $A^{\alpha+\beta} = A^{\alpha}A^{\beta}$
5. $(AB)^{\alpha} = A^{\alpha}B^{\alpha}$
6. $(A^{\alpha})^{\beta} = A^{\alpha\beta}$
7. $A^1 = A$

It is also assumed that positive numbers ($a, b, c \ldots$) also belong to Π and that their power a^{α} are calculated as usually. Thus the positive numbers can be considered as subspace Π^0 of Π (satisfying the same axioms as Π).
We can say that any element of Π which does not belong to Π^0, i.e which is not a number, will be called a **dimensional quantity**. Above axioms for the **dimensional space** are fully analogical to the axioms of linear (vector) space where multiplication of elements of dimensional space AB is replaces by the sum $A + B$ and power rising A^{α} by αA.
The algebraic dimension (rank) of dimensional space Π is 3. It stems from the fact that we used three fundamental measure units: kilogram, meter and seconds (MLT). This fundamental dimensions (MLT) we can consider as fundamental or elementary basis for Π space. By analogy to the linear vector space where elementary basis are $\mathrm{e}_1 = (1,0,0),\ \mathrm{e}_2 = (0,2,0),\ \mathrm{e}_3 = (0,0,3)$.

Definition 1. *The elements A_1, A_2, A_3 of Π will be called dimensionally independent when the equality*

$$A_1^{\alpha_1} A_2^{\alpha_2} A_3^{\alpha_3} = a \tag{1}$$

where a is a real number, hold if and only if when $\alpha_1 = \alpha_2 = \alpha_3 = 0$ (and $a = 1$)

On the basis of this definition we can formulate simply criterion for the set of variables which are dimensionally **independent**. All quantities A_i, due to fact that they are belong to dimensional space Π, can be expressed by elements of fundamental basis:

$$A_i = a_i M^{\beta_{i1}} L^{\beta_{i2}} T^{\beta_{i3}} \tag{2}$$

In the term of fundamental basis, equation (1) can be rewrite as:

$$\left(M^{\beta_{11}} L^{\beta_{12}} T^{\beta_{13}}\right)^{\alpha_1} \left(M^{\beta_{21}} L^{\beta_{22}} T^{\beta_{23}}\right)^{\alpha_2} \left(M^{\beta_{31}} L^{\beta_{32}} T^{\beta_{33}}\right)^{\alpha_3} = M^0 L^0 T^0 \tag{3}$$

Comparing the exponents with the same basis we obtain

$$\begin{aligned} \beta_{11}\alpha_1 + \beta_{21}\alpha_2 + \beta_{31}\alpha_3 &= 0 \\ \beta_{12}\alpha_1 + \beta_{22}\alpha_2 + \beta_{32}\alpha_3 &= 0 \\ \beta_{13}\alpha_1 + \beta_{23}\alpha_2 + \beta_{33}\alpha_3 &= 0 \end{aligned}$$

or in more compact form as

$$\begin{bmatrix} \beta_{11} & \beta_{21} & \beta_{31} \\ \beta_{12} & \beta_{22} & \beta_{31} \\ \beta_{13} & \beta_{32} & \beta_{33} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

From algebra we know that the system (4) has a unique, in this case zero solution $\alpha_i = 0$, when determinant of the algebraic linear system (4) is not equal zero

$$det \begin{bmatrix} \beta_{11} & \beta_{21} & \beta_{31} \\ \beta_{12} & \beta_{22} & \beta_{31} \\ \beta_{13} & \beta_{32} & \beta_{33} \end{bmatrix} \neq 0 \tag{5}$$

So the equation (5) determines the criterion for the dimensions of quantities A_1, A_2, A_3 to be the dimensionally independent.

Example 1. *Let as check if the quantities: velocity v , density ρ, and diameter D are dimensionally independent. At first we must build up the following matrix*

	v	ρ	D
M	*0*	*1*	*0*
L	*1*	*-3*	*1*
T	*-1*	*0*	*0*

where we have written the variables v, ρ,D on the top and in vertical column underneath the exponents one needs to express them in elementary basis MLT . For example $[v] = M^0 L^1 T^{-1}$. *Above array is called a* **dimensional matrix**. *It is not difficult to calculate that determinant is different from zero* ($det|\beta_{ij}| = -1$)*, and we can conclude that the variables* (v, ρ, D) *are dimensionally independent.*

> *Any three dimensionally independent* (A_1, A_2, A_3) variables one can regards as a *dimensional basis*.

3 Buckingham's Π–theorem

The Pi–theorem is based on the rule of ***dimensional homogeneity***.

> If an equation truly express a proper relationship between variables in physical process, it will be dimensionally homogeneous; i. e each of its additive terms will have the same dimensions.

Consider the relation which express the displacement of falling body

$$S = S_0 + V_0 t + \frac{1}{2} g t^2 \tag{6}$$

Each term in this equation is a displacement, or length, and has dimension [L]. The equation is dimensionally homogeneous. Consider Bernoulli's equation for incompressible flow

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = const \tag{7}$$

Each term, including the constant, has dimension of length [L]. The equation is dimensionally homogeneous and gives proper results for any consistent set of units.
One can deduced from the physical property that the ratio of two distinct values of the same derived quantity is independent of scale used.
The Buckingam's Pi theorem give us a way of the building the relation between the dimensional variables.

Theorem 1. *Let us assume that in physical experiment one has* $y = f(x_1, x_2, x_3, \ldots, x_n)$ *and let assume that the first three variables are dimensionally independent, then the function may be reorganized into form:*

$$y = \varphi(\pi_1, \pi_2, \ldots, \pi_{n-3}) x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \tag{8}$$

where

$$\pi_1 = \frac{x_4}{x_1^{\beta_{11}} x_2^{\beta_{12}} x_3^{\beta_{13}}}, \qquad \pi_2 = \frac{x_5}{x_1^{\beta_{21}} x_2^{\beta_{22}} x_3^{\beta_{23}}}, \ldots \qquad \pi_{n-3} = \frac{x_n}{x_1^{\beta_{(n-3)1}} x_2^{\beta_{(n-3)2}} x_3^{\beta_{(n-3)3}}} \tag{9}$$

Remark 1. *It should be clear from the presentation above that due to fact that we assume that* (x_1,x_2,x_3) *are dimensionally independent we can used these variables as the dimensional basis. The rest of the variables* $(x_4,x_5,\dots x_n) \in \Pi$ *one can express with the help of this basis. The* $\varphi(\pi_1,\pi_2,\dots,\pi_{n-3})$ *is non-dimensional and belongs to subspace* Π^0. *It is worth to notice that number of independent variable in* φ *was reduced by 3 (it is by the rank of algebraic dimension of dimensional space* Π*). The dimension of monomial* $[x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}]$ *in front of* φ *gives the dimension of* $[y]$ *variable.*

When one divided the equation (8) by side by $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}$ its results in

$$\frac{y}{x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}} = \pi = \varphi(\pi_1,\pi_2,\dots,\pi_{n-3}) \tag{10}$$

The relation (10) is fundamental in the planning of the physical experiment. The pi theorem is a formal method of forming dimensionless groups π_i. We must remember that **dimensional analysis** gives us only information about the general form of a relation between some unknown and other variables in physical problem. It does not determine the exact form of his relation, which must be found either by solving mathematical equations governing the problem or by measurements of the unknown.

Example 2. *Under laminar conditions, the volume flow* q *through a tube with radius* R *and length* l *is a function of viscosity* μ*, pressure drop per unit length* $\Delta p/l$

$$q = f(\frac{\Delta p}{l},\mu,R). \tag{11}$$

Using the Π*–theorem, rewrite this relation in dimensionless form. How does the volume flow change if the radius of the pipe is tripled?*
Solution. *Due to fact that number of the variables on right side of (11) is three then the amount of non–dimensional variables* π *will be* $n-3=3-3=0$. *The relation takes a form*

$$q = c\left(\frac{\Delta p}{l}\right)^{\alpha_1} \mu^{\alpha_2}\, R^{\alpha_3} \tag{12}$$

where c *is a real number. We have to check if the variables* $(\frac{\Delta p}{l},\mu,R)$ *are dimensionally independent. We must build the dimensional matrix:*

	$\Delta p/l$	μ	R
M	1	1	0
L	-2	-1	1
T	-2	-1	0

Determinant of the dimensional matrix is not equal zero (is equal -1), so $(\frac{\Delta p}{l},\mu,R)$ *are dimensionally independent and one can use them as a basis. To find the value of the exponents* ($\alpha_i, i=1,2,3$) *we express each variables in equation (12) by elementary dimensional basis* (MLT). *Namely* $[q] = M^0\,L^3\,T^{-1}$ *and equation (12) can be rewrite as follows:*

$$M^0\,L^3\,T^{-1} = \left(M^1L^{-2}T^{-2}\right)^{\alpha_1}\,\left(M^1L^{-1}T^{-1}\right)^{\alpha_2}\,\left(M^0L^1T^0\right)^{\alpha_3} \tag{13}$$

Equality of the nominal with the same basis (MLT) require the equality of the exponents. One obtained system of algebraic equation:

$$0 = \alpha_1 + \alpha_2 \tag{14}$$
$$3 = -2\alpha_1 - \alpha_2 + \alpha_3 \tag{15}$$
$$-1 = -2\alpha_1 - \alpha_2 \tag{16}$$

Solutions are: $\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 4$. Equation(12) takes the form:

$$q = c(\frac{\Delta p}{l})\, \frac{1}{\mu}\, R^4 \tag{17}$$

It is ***Hagen-Poiseuille's*** *low which we knew from lecture* **n5**. *The constant c is equal $c = \pi/8$. To determine this constant from experiment we need only the one measurement of q for given radius R, drop off pressure $\Delta p/l$ and viscosity μ. When the radius of the pipe is tripled the flow rate increase* 81 *times*

Example 3. *Consider the case of drag on a sphere of diameter d moving at a speed U trough a fluid of density ρ and viscosity μ. The drag force can be written as*

$$D = f(d, U, \rho, \mu) \tag{18}$$

If we do not use dimensional analysis, we would have to conduct an experiment to determine force D vs. d, keeping U,ρ and μ fixed. we would then have to conduct an experiment to determine D as a function of U, keeping d,ρ,and μ fixed and so on.
At first we will create the dimensional matrix

	d	U	ρ	μ
M	0	0	1	1
L	1	1	-3	-1
T	0	-1	0	-1

Now the number of variables is grater than algebraic dimension of Π space. In such a case we should choose from the dimensional matrix the submatrix with rank 3. One can check that now we can choose three set of independent variable $\{d, U, \rho\}$, $\{d, U, \mu\}$ and $\{U, \rho, \mu\}$ that can be used as a dimensional basis. All of these basis are valid. Dimensional analysis do not say which one is the best. Choosing one specific set of independent variables as the dimensional basis depend on the intuition of researcher or the historical tradition. Sometimes some of them are more comfortable in study than the others. Further, in this example it was taken the set $\{d, U, \rho\}$. By virtu of Π–theorem relation (18) take form:

$$D = \varphi(\pi) d^{\alpha_1} U^{\alpha_2} \rho^{\alpha_3} \tag{19}$$

Expressing the dimensional variable by the elementary dimensional basis (fundamental units) $\{MLT\}$, $[D] = M^1 L^1 T^{-2}$ one obtain

$$M^1 L^1 T^{-2} = \left(M^0 L^1 T^0\right)^{\alpha_1} \left(M^0 L^1 T^{-1}\right)^{\alpha_2} \left(M^1 L^{-3} T^0\right)^{\alpha_3} \tag{20}$$

lead to the system of the equation

$$1 = \alpha_3 \tag{21}$$
$$1 = \alpha_1 + \alpha_2 - 3\alpha_3 \tag{22}$$
$$-2 = -\alpha_2 \tag{23}$$

Solution to above system of algebraic equation is : $\alpha_1 = 2, \alpha_2 = 2, \alpha_3 = 1$

The π *variable has form:*

$$\pi = \frac{\mu}{d^{\beta_1} U^{\beta_2} \rho^{\beta_3}}$$

Applying the methodology like in equation (20) yields $\beta_1 = 1, \beta_2 = 1, \beta_3 = 1$. *The relation (19) takes the form:*

$$D = \varphi\left(\frac{\mu}{d\ U\ \rho}\right) d^2 U^2 \rho \tag{24}$$

The $\frac{\mu}{d\ U\ \rho} = \frac{1}{Re}$ *we can regarded that the* φ *depend on Reynolds number* $Re = \frac{Ud}{\nu}$. *Now the (24) may be rewrite as follow*

$$\frac{D}{d^2 U^2 \rho} = \varphi(Re) \tag{25}$$

In practice, the left side of Eq. (25) is called as drag coefficient and is defined as

$$c_x = \frac{D}{\frac{1}{2} A U^2 \rho}$$

where A means a frontal area. For a sphere $A = d^2$. *A dimensional analysis of equation (18) reduced independent variables to one in (24), and consequently a single experimental curve* $c_x = \varphi(Re)$ *(see fig. 1). Not only the presentation of data is united and simplified, the cost of experimentation is drastically reduced. It is clear that we need not vary the the fluid viscosity or density at all; we could obtain all the data of Figure 1 in one wind tunnel experiment in which we determine the force D for various values of* U. *However, if we want to find the drag force for a fluid of different density or viscosity, we can still use Figure 1. Note that the Reynolds number in Eq. (25) is written as the independent variable because it can be externally controlled in an experiment and the drag coefficient is written as a dependent variable.*

Example 4. *In the flow of fluid trough a long cylindrical pipe, the pressure drop per unit length of pipe* $\Delta p/l$ *is completely determined by the mean fluid velocity* U, *diameter of the pipe* D *the fluid density* ρ, *fluid viscosity* μ *and absolute roughness of the pipe* ε. *Use dimensional analysis to determine the general form of the equation*

$$\frac{\Delta p}{l} = f(D, U, \rho, \mu, \varepsilon) \tag{26}$$

From Π *theorem we expect that amount of non-dimensional* π *- variable will be* $n - 3 = 5 - 3 = 2$. *We start from dimensional matrix trying to find out the dimensional basis.*

	d	U	ρ	μ	ε
M	0	0	1	1	0
L	1	1	-3	-1	1
T	0	-1	0	-1	0

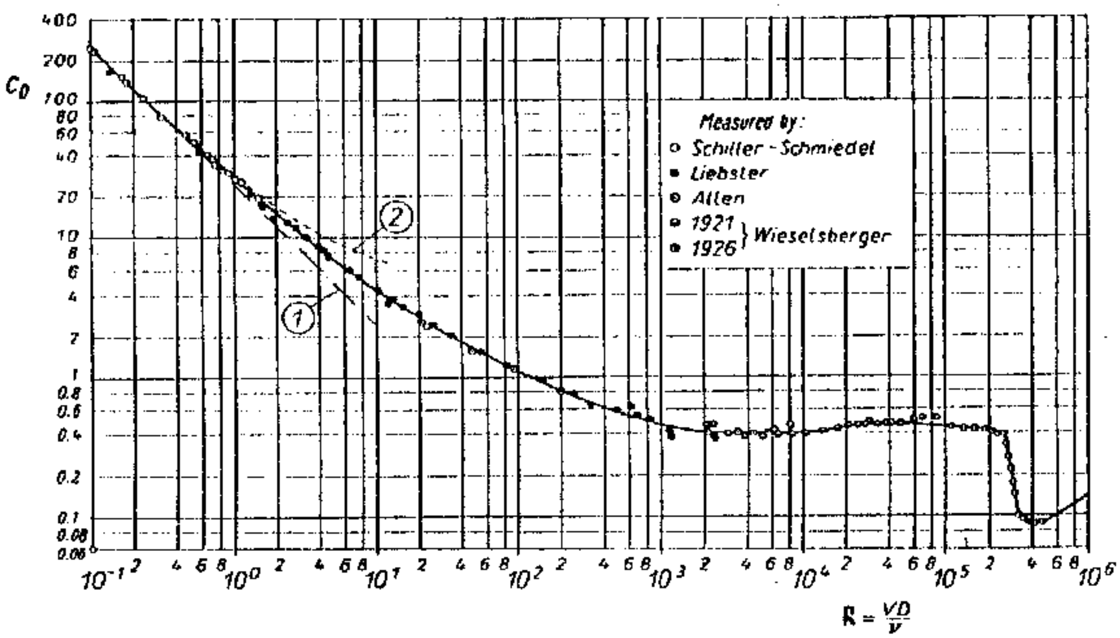

Figure 1: Drag coefficient for the a sphere as function of Reynolds number. The characteristic (frontal) area is taken as $A = \pi d^2/4$. The reason for the sudden drop of C_D,called as drag crasis, at $Re \sim 3 \cdot 10^5$ is the transition of the laminar boundary layer to a turbulent one. Curve (1)– Stokes's theory, curve(2)–Oseen's theory

We can find a 4 sub matrixes with rank 3: $\{D,U,\rho\}$, $\{D,\rho,\mu\}$, $\{U,\rho,\mu\}$, $\{U,\rho,\varepsilon\}$. *In further calculation we choose* $\{d,U,\rho\}$. *By virtu of* Π *theorem the relation of (26) can can write*

$$\frac{\Delta p}{l} = \varphi(\pi_1,\pi_2) D^{\alpha_1} U^{\alpha_1} \rho^{\alpha_3}, \tag{27}$$

$$\textit{where} \qquad \pi_1 = \frac{\mu}{D^{\beta_{11}} U^{\beta_{12}} \rho^{\beta_{13}}} \qquad \pi_2 = \frac{\varepsilon}{D^{\beta_{21}} U^{\beta_{22}} \rho^{\beta_{23}}} \tag{28}$$

From the previous example we known that $\pi_1 = \frac{1}{Re}$, $\quad Re = \frac{Ud\rho}{\mu}$. *It is easy the check that* $\pi_2 = \frac{\varepsilon}{d}$. *To determine the* α_i *we applied the procedure as in previous examples* $[\Delta p/l] = M^1 L^{-2} T^{-2} = \left(M^0 L^1 T^0\right)^{\alpha_1} \left(M^0 L^1 T^{-1}\right)^{\alpha_2} \left(M^1 L^{-3} T^0\right)^{\alpha_3}$. *Comparing the powers of monomial with the same base we obtain the system of equation*

$$1 = \alpha_3 \tag{29}$$

$$-2 = \alpha_1 + \alpha_2 - 3\alpha_3 \tag{30}$$

$$-2 = -\alpha_2 \tag{31}$$

Solution of the above system is $\alpha_1 = -1, \alpha_2 = 2, \alpha_3 = 1$. *The relation (27) take the form*

$$\frac{\Delta p}{l} = \varphi(Re, \frac{\varepsilon}{D}) D^{-1} U^2 \rho^2 \tag{32}$$

It is a famous Darcy-Weisbach equation. Let as recall that Darcy–Weisbach equation has form

$$\boxed{h_l = \frac{\Delta p}{\rho g} = f\ \frac{l}{D}\ \frac{v^2}{2g}} \tag{33}$$

When we compare (33) with (32) that this formula are the same up to the number 2, $f \equiv \varphi(Re, \frac{\varepsilon}{D}$. *The dimensional analysis is unable to be so accurate. Ones again pay attention to the huge simplification of the problem. We reduced the number of the independent variable from 5 in (26) to two, and eventually to the family of curves* $\varphi(Re)|_{\varepsilon})$, *that are parameterized by* relative roughness $\varepsilon = \frac{\varepsilon}{D}$ *(see fig.2)*

$$\frac{\Delta p}{lD^{-1}U^2\rho^2} = \varphi(Re, \frac{\varepsilon}{D}) \tag{34}$$

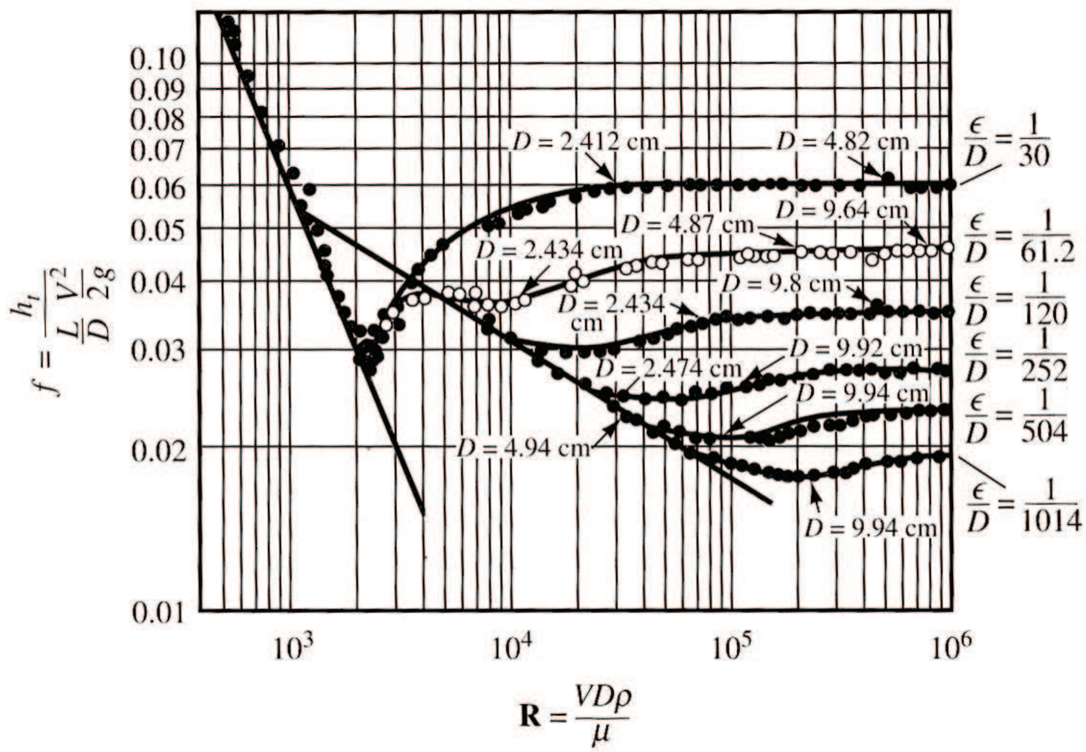

Figure 2: Nikuradse's sand-roughened-pipe tests. Nikuradse used three sizes of pipes and glued sand grains (ε = diameter of the sand grains) of practically constant size to the interior walls so that he had the same values of ε/D for different pipes.

4 Problems

1. Derive an expression for the period of pendulum T=f(m,L,g). The parameter are the length of the pendulum L, mass of the bob m and gravity g

2. A weir is an obstruction in channel flow that can be calibrated to measure flow rate. The value flow q varies with gravity g, upstream water height H above the weir and density of water ρ; $q = f(\rho, g, h)$. Find a unique functional relationship.

3. The torque M on an axial-flow turbine is a function of fluid density ρ, rotor diameter D, angular rotation rate Ω and fluid flow q. Find the functional relationship using the Π theorem, $M = f(\rho, D, \Omega, q)$. If it is known that M is proportional to q for a particular turbine, hao would M vary with Ω and D for the turbine?

4. For the wall layer, Prandtl deduced in 1930 that velocity u in boundary layer must be independent of the shear layer thickness and depends on viscosity μ, share stress on the wall τ_w, density ρ and distance form the wall y: $u = f(\mu, \tau, \rho, y)$. Find functional relationship using Π theorem.

5. Karman in 1933 deduced that u in the outer layer is independent of molecular viscosity, but its deviation from the stream velocity U must depends on layer thickness δ, share stress on the wall τ_w, density ρ, and distance from the wall y;$\Delta u = U - u = g(\delta, \tau, \rho, y)$. Derive the functional relationship in the form $\frac{U-u}{u^*} = G(\frac{y}{\delta})$, where the quantity u^* is termed the fiction velocity $u^* = \left(\frac{\tau_w}{\rho}\right)^{1/2}$

6 . On a fluid rotated as a solid about a vertical axis with angular velocity ω, pressure p in radial direction depends upon speed ω, radius r, and fluid density ρ. Obtain the form of equation for Δp, $\Delta p = f(\omega, r, \rho)$

7 . The size of droplets produced by a liquid spray nozzle is thought to depend upon the nozzle diameter Dm jet velocity U, and properties of the liquid ρ,μ, and σ: $d = f(D, U, \rho, \mu, \sigma)$. Rewrite this relation in dimensionless form.

8. Derive the expression for the drag on a submerged torpedo. The parameters are the size of the torpedo L, the velocity of the torpedo V,the viscosity of the water μ, and the density of the water ρ: $F_D = f(L, V, \rho, \mu)$.
During this course I will be used the following books:

References

[1] F. M. White, 1999. *Fluid Mechanics*, McGraw-Hill.

[2] B. R. Munson, D.F Young and T. H. Okiisshi, 1998. *Fundamentals of Fluid Mechanics*, John Wiley and Sons, Inc. .

[3] Y. Nakayama and R.F. Boucher, 1999.*Intoduction to Fluid Mechanics*, Butterworth Heinemann.

[4] Y.A. Cengel and J. M. Cimbala, 2006. *Fluid Mechanics*, McGraw Hill.

[5] J.M. McDonough, 2004. *Lectures in Elementary Fluid Dynamics: Physics, Mathematics and Applications*, University of Kentucky, Lexington.