1 Balance of Momentum

In particle mechanics a particle $P$ that has mass $m$ and move with the velocity $v$ is said to posses (linear) momentum $mv$ and the general statement of Newton’s second law is

$$\frac{d(mv)}{dt} = F. \quad (1)$$

The term $F$ encompass the ”sum of the forces acting on the mass “. To generalize this law to continuum mechanics we define the linear momentum at time $t$ possed by material of density $\rho(x,t)$ that occupies a region $\Omega(t)$ as

$$\int_{\Omega(t)} \rho(x,t)v \, dv.$$  

In fluid mechanics we regarded two main types of forces:

1. body forces $F_B$ acting on the entire region $\Omega(t)$ for ex. in gravitational field density of this force is $f = (0, 0, -g)$ and

$$F_B = \int_{\Omega(t)} \rho(x,t)f \, dv$$
2. surface forces $F_S$, short length forces, which act across the surface. They are described by the vector of tension $t(x,t)$ and surface force is given by

$$F_S = \int_S t \, dS.$$

**Definition 1.** Let $t(x,t,n)$ be given tension vector depending on time $t$, position $x$, and unit vector $n$. Let velocity vector $v$ and $\rho$ be positive fraction and $f$ a given vector field. We say that $(v, \rho, f, t)$ satisfy the balance of momentum principle if

$$\frac{d}{dt} \int_{\Omega(t)} \rho v \, dv = \int_{\Omega(t)} \rho f \, dv + \int_S t \, dS \tag{2}$$

where $S$ is boundary of $\Omega(t)$.

## 2 One-dimensional equation of motion. Bernoulli’s equation

General flows are three dimensional, but many of them may be studied as if they are one dimensional. For example, whenever a flow in a tube is considered, if it is studied in terms of mean velocity, it is a one-dimensional flow which is studied very simply. Let us applied the principle of conservation of momentum (2) to the infinitesimal, cylindrical element of fluid having the cross-sectional area $dA$ and length $ds$ which lay along the streamline. We assume that tension $t$ is described by the pressure $p$, $t = -p \cdot n$. It means the tension $t$ is always orthogonal to the surface. Such tension is proper only for ideal (invicid) fluid. We also assume that the flow is stationary, it is that all local time derivatives are equal to zero $\frac{\partial v}{\partial t} = 0$. Around a cylindrical element of fluid having the cross-sectional area $dA$ and length $ds$ is considered. Let $p$ be the pressure acting on the lower face, and pressure $p + \frac{\partial p}{\partial s} ds$ acts on the upper face a distance $ds$ away. The gravitational force acting on this element is its weight, $pgdAd s$. Applying Newton’s second law of motion(1) to this element, the resultant force acting on it, and producing acceleration along the streamline

![Figure 1: Force acting on fluid on streamline](image-url)
$(z(s), x(s))$ is the force due to the pressure difference across the streamline and the component of any other external force (in this case only the gravitational force) along the streamline. Therefore the following equation is obtained:

$$
\rho \ dA \ ds \ \frac{dv}{dt} = -dA \ \frac{\partial p}{\partial s} \ ds = \rho \ g \ dA \ ds \ \cos(\Theta)
$$

Recalling that

$$
\frac{dv}{dt} = \frac{\partial v}{\partial t} + v \ \frac{\partial v}{\partial s}, \quad \cos(\Theta) = \frac{\partial z}{\partial s}
$$

equation (3) can be rewritten

$$
\frac{\partial v}{\partial t} + v \ \frac{\partial v}{\partial s} = -\frac{1}{\rho} \ \frac{\partial p}{\partial s} - g \ \frac{\partial z}{\partial s}
$$

Equation (4) is called Euler’s equation of motion for one-dimensional non-viscous fluid flow. More exactly it is a projection of the momentum equation on the direction of streamline. In incompressible fluid flow with two unknowns (v and p), equation (4) and the continuity equation $Av = \text{const}$ must be solved simultaneously. Further we will assume that flow is steady $\frac{\partial v}{\partial t} = 0$ and fluid is incompressible with constant density $\rho = \text{const}$. One can noticed that term $v \ \frac{\partial v}{\partial s}$ can express as $\frac{1}{2} \ \frac{\partial}{\partial s} (v^2)$. Taking all terms of the equation (4) to the one side, the equation (4) can be rewrite as

$$
\frac{\partial}{\partial s} \left( \frac{1}{2} v^2 + \frac{p}{\rho} + gz \right) = 0
$$

From the equation (6) is clear that along streamlines the

$$
\frac{v^2}{2} + \frac{p}{\rho} + gz = \text{const}. \quad (6)
$$

Equation (6) is Bernoulli equation. We recognize that $\frac{v^2}{2}$ as kinetic energy, $gz$ as potential energy, and $\frac{p}{\rho}$ as flow energy, all per unit mass. Therefore the Bernoulli equation can be viewed as "conservation of mechanical energy principle".

The sum of the kinetic, potential, and flow energies of a fluid particle is constant along a streamline during steady flow when the compressibility and frictional effects are negligible

Despite the highly restrictive approximations used in its derivation, the Bernoulli equation is commonly used in practice since a variety of practical fluid flow problems (steady and incompressible with negligible friction forces) can be analyzed to reasonable accuracy with it.

It is often convenient to represent the level of mechanical energy graphically using heights to facilitate visualization of various terms of the Bernoulli equation. This is done by dividing each term of Bernoulli equation by $g$ to give

$$
\frac{v^2}{2g} + \frac{p}{\rho g} + z = H = \text{const}. \quad (7)
$$

Each term in this equation has the dimension of length and represent some kind of "head" of a flowing fluid as follows:
- $H$ is **total head** of flow
- $\frac{p}{\rho g}$ is **pressure head**; it represents the height of a fluid column that produces the static pressure $p$
- $\frac{v^2}{2g}$ is **velocity head**; it represents the elevation needed for a fluid to reach the velocity $v$ during frictionless free fall.
- $z$ is the **elevation head**; it represents the potential energy of fluid.

The value of the constant can be evaluated at any point on the streamline where the pressure, density, velocity and elevation are known. The Bernoulli equation can be written between any two points on the same streamline as

$$\frac{v_1^2}{2g} + \frac{p_1}{\rho g} + z_1 = \frac{v_2^2}{2g} + \frac{p_2}{\rho g} + z_2.$$  \hspace{1cm} (8)

![Figure 2: Exchange between pressure head and velocity head](image)

### 3 Application of Bernoulli equation

Various problems on the one-dimensional flow of an ideal fluid can be solved by jointly using Bernoulli’s theorem and the continuity equations.

#### 3.1 Velocity Measurement by a Pitot Tube

A useful concept associated with the Bernoulli equation deals with the stagnation and dynamic pressure. The pressure arises from conversion of kinetic energy into “pressure increment” due to fact that a fluid is brought to the rest at stagnation point.

#### 3.2 Flow Rate Measurement by a Venturi tube

An effective way to measure the flowrate through a pipe is to place some type of restriction within the pipe (see figure) and to measure the pressure difference between the low-velocity, high-pressure upstream section(1), and the high-velocity, low-pressure downstream section(2).
3.3 Flow through a small hole

We study the case where water is discharging from a small hole on the side of a water tank. Such a hole is called an orifice.

3.4 Flow in a syphon

A syphon is a device that can be used to transfer liquids between two containers without the use of a pump. During this course I will be using the following books:

References


Figure 3: Pitot tube. The static pressure, dynamic pressure and total pressure in a stagnation point

\[ V = \sqrt{\frac{2(P_{\text{stag}} - P)}{\rho}} \]

Assuming that pipe line is horizontal, \( z_1 = z_2 \)

\[ \frac{v_1^2}{2g} + \frac{p_1}{\rho g} + z_1 = \frac{v_2^2}{2g} + \frac{p_2}{\rho g} + z_2. \]

From the continuity equation, \( v_1 = v_2 \frac{A_2}{A_1} \), therefore

\[ q_v = v_2 A_2 = \frac{A_2}{\sqrt{1 - (A_2/A_1)^2}} \sqrt{2g \frac{p_1 - p_2}{\rho g}}, \]

\[ \frac{p_1 - p_2}{\rho g} = H \]

Figure 4: Venturi tube
\[ \frac{v_A^2}{2g} + \frac{p_A}{\rho g} + z_A = \frac{v_B^2}{2g} + \frac{p_A}{\rho g} + z_B. \]

Assuming that the water tank is large and the water level does not change, at point \( A \), \( v_A = 0, z_A = H, z_B = 0, p_A \) is the atmospheric pressure, then

\[ \frac{p_A}{\rho g} + H = \frac{p_A}{\rho g} + \frac{v_B^2}{2g} \]

\[ v_B = \sqrt{2gH} \]

Equation is called as Torricelli’s equation.

We write Bernoulli’s equation between location 1 and 2.

\[ \frac{v_1^2}{2g} + \frac{p_1}{\rho g} + y_1 = \frac{v_2^2}{2g} + \frac{p_2}{\rho g} + y_2. \]

But from the figure, \( y_1 = y_2 \), and we assume that \( v_1 \) can be neglected. Since \( p_1 = p_{atm} \) this results in

\[ \frac{v_2^2}{2g} = \frac{p_{atm} - p_2}{\rho g} \]

Next, we write Bernoulli’s equation for the point 2 and 3, and from continuity equation we know that \( v_2 = v_3 \),

\[ \frac{p_2 - p_{atm}}{\rho g} = y_3 - y_2 \]

\[ v_2 = \sqrt{2g(y_2 - y_3)} \]