Kinematics of fluid motion

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Contents

Kinematics - analysis of fluid motion, that is description of motion in terms of displacement, velocity and acceleration but without regard to the forces causing it.

1 Kinematical preliminaries

Fluid flow is a intuitive physical motion which is represented mathematically by continues transformation of three-dimensional Euclidean space into itself $\mathbb{R}^3 \to \mathbb{R}^3$. Parameter t describing the transformation is identified with the time and we may suppose its range to be $-\infty < t < \infty$. Each particle of fluid flows follows a certain *trajectory* (see fig.1). Thus

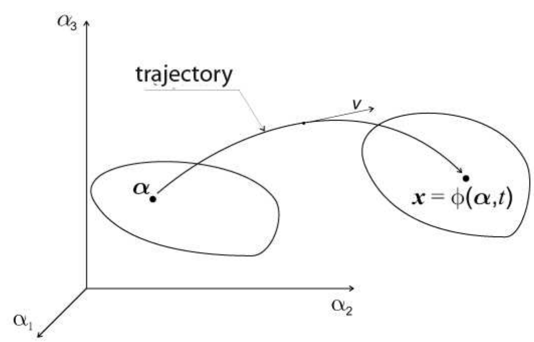

Figure 1: Motion of fluid particles

for each $\boldsymbol{\alpha} \subset \Omega(t = 0)$ there is a path $\mathbf{x}(t) = \mathbf{\Phi}(\boldsymbol{\alpha}, t)$ representing the trajectory of $\boldsymbol{\alpha}$.

In order to label the different trajectories, let us write $\Phi(\alpha, t)$ for the path followed by $\boldsymbol{\alpha}$ with the initial condition $\mathbf{\Phi}(\boldsymbol{\alpha}, 0) = \boldsymbol{\alpha}$

$$\mathbf{x} = \mathbf{\Phi}(\boldsymbol{\alpha}, t). \tag{1}$$

The mapping equation (1) specifies the *path* of particle P initially at $\boldsymbol{\alpha}$; on the other hand, for fixed t equation (1) determines a transformation of the region initially occupied by the fluid into its position at time t.

We assume that initially distinct points remains distinct throughout the entire motion, or, in the other words, the transformation (1) possesses an inverse.

$$\boldsymbol{\alpha} = \Phi^{-1}(\mathbf{x}). \tag{2}$$

The Jacobian of the transformation (1)

$$J = \mathbf{det} \begin{bmatrix} \dfrac{\partial \Phi_1(\boldsymbol{\alpha}, t)}{\partial \alpha_1} & \dfrac{\partial \Phi_1(\boldsymbol{\alpha}, t)}{\partial \alpha_2} & \dfrac{\partial \Phi_1(\boldsymbol{\alpha}, t)}{\partial \alpha_3} \\ \frac{\partial \Phi_2(\boldsymbol{\alpha}, t)}{\partial \alpha_1} & \dfrac{\partial \Phi_2(\boldsymbol{\alpha}, t)}{\partial \alpha_2} & \dfrac{\partial \Phi_2(\boldsymbol{\alpha}, t)}{\partial \alpha_3} \\ \frac{\partial \Phi_3(\boldsymbol{\alpha}, t)}{\partial c_1} & \dfrac{\partial \Phi_3(\boldsymbol{\alpha}, t)}{\partial \alpha_2} & \dfrac{\partial \Phi_3(\boldsymbol{\alpha}, t)}{\partial \alpha_3} \end{bmatrix} \tag{3}$$

must be different form zero:

$$0 < J < \infty. \tag{4}$$

1.1 Lagrangian variables for the description of the motion

The variable $(\boldsymbol{\alpha}, t)$ which single out individual particles is called *material variables* or *Lagrangian variables*. The *velocity* of particle (in Lagrangian variable)and *acceleration* is given by the definition:

$$V_L = \frac{\partial \mathbf{\Phi}(\boldsymbol{\alpha}, t)}{\partial t}, \qquad a_L = \frac{\partial^2 \mathbf{\Phi}(\boldsymbol{\alpha}, t)}{\partial t^2} \tag{5}$$

1.2 Eulerian variables for the description of the motion

Although a flow is completely determined by the transformation(1), it is also important to consider the state of motion at a given point during the course of time. This is descrived by functions

$$\varrho = \varrho(\mathbf{x}, t), \qquad \mathbf{v} = v(\mathbf{x}, t) \tag{6}$$

which give the density and velocity , etc., of particle which happens to be at the position $\mathbf{x}$ at time t. The variables $(\mathbf{x}, t)$ used in the field description of the flow is called *spacial* *variables* or *Eulerian variables*. By means of equation (1) any quantity f which is a function of spatial variable $(\mathbf{x}, t)$ is also a function of the material variables $(\boldsymbol{\alpha}, t)$, and conversely.

$$f(\mathbf{x}, t) = f(\boldsymbol{\Phi}(\boldsymbol{\alpha}, t), t) \tag{7}$$

Time derivative of $f(\mathbf{x}, t)$ is express by the formula

$$\frac{df(\mathbf{x}, t)}{dt} = \frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(\frac{\partial f}{\partial x_i} \frac{d\ x_i}{d\ t} \right) \tag{8}$$

The expression (8) is called *substantial derivative*($\frac{\partial}{\partial t}$ is called as a local, and $\mathbf{v}\nabla f$ as a convective). Acceleration of fluid particle expressed in Eulerian variables:

$$\frac{\mathrm{d}v_i(\mathbf{x}, t)}{\mathrm{dt}} = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_i} v_i \tag{9}$$

2 Fundamentals of flow visualization

Definition 1 *A* ***pathline*** *is the actual path traveled by an individual fluid particle over some time period*

$$\mathbf{x}(t) = \boldsymbol{\Phi}(\boldsymbol{\alpha}, t) \tag{10}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} \tag{11}$$

Definition 2 *A* ***streamline*** *is a curve that is everywhere tangent to the instantaneous local velocity vector* $\mathbf{v} = (u, v, w)$

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Definition 3 *A* ***streakline*** *is the locus of fluid particles that have passed sequentially through a prescribed point in the flow*

During this course I will be used the following books:

References

[1] F. M. White, 1999. *Fluid Mechanics*, McGraw-Hill.

[2] B. R. Munson, D.F Young and T. H. Okiisshi, 1998. *Fundamentals of Fluid Mechanics*, John Wiley and Sons, Inc. .

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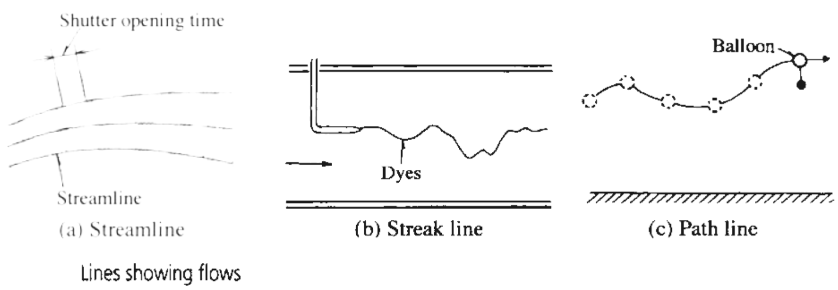

Figure 2: Pathline,stremline,streakline