Principle of Linear Momentum Application

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1 Introduction

Let as recall the material from the lecture *n3 equation of motion.* Equation of motions is build on the basis of the balance of momentum principle. In particle mechanics a particle P that has mass m and move with the velocity v is said to posses (linear) momentum mv and the general statement of Newton 's second law is

$$\frac{d(mv)}{dt} = F. \tag{1}$$

The term F encompass the "sum of the forces acting on the mass ". To generalize this law to continuum mechanics we define the linear momentum at time t possed by material of density $\rho(\mathbf{x},t)$ that occupies a region $\Omega(t)$ as

$$\int_{\Omega(t)} \rho(\mathbf{x},t)\mathbf{v}\, d\upsilon.$$

This region $\Omega(t)$ is moved with the fluid and boundary of this region is created all the time by the same fluid particles. The mass enclosed in this region is constant. Such a region is called the **closed system** In fluid mechanics we regarded two main types of forces:

1. body forces F_B acting on the entire region $\Omega(t)$ for e.g in gravitational field density of this force is $\mathbf{f} = (0,0,-g)$ and

$$F_B = \int_{\Omega(t)} \rho(\mathbf{x},t)\mathbf{f} d\upsilon$$

2. surface forces F_S, short length forces, which act across the surface. They are descried by the vector of tension $\mathbf{t}(\mathbf{x},t)$ and surface force is given by

$$F_S = \int_S \mathbf{t}\, dS.$$

In this lecture we will assumed that surface forces are represent by pressure. Pressure forces are isotropic, i.e direction-independent. This means that are described by a scalar filed $p(x,t)$ such that the force exerted across an arbitrarily-oriented surface element δS at $\mathbf{x}$ with unit normal $\mathbf{n}$ is always $\mathbf{n}p\delta S$, independent of the chosen orientation.

Definition 1. *Let $\mathbf{t}(\mathbf{x},t,\mathbf{n})$ be given tension vector depending on time t, position $\mathbf{x}$, and unit vector $\mathbf{n}$. Let velocity vector $\mathbf{u}$ and $\rho > 0$ be positive faction and $\mathbf{f}$ a given vector field. We say that $(\mathbf{u},\rho,\mathbf{f},\mathbf{t})$ satisfy the balance of momentum principle if*

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\Omega(t)} \rho\mathbf{u}d\upsilon = \int_{\Omega(t)} \rho\mathbf{f}d\upsilon + \int_S \mathbf{t}dS \tag{2}$$

where S is boundary of $\Omega(t)$.

Applying the **Reynolds Transport Theorem** to the component ρu_i (see lecture *n3 equation of motion*) we can rewrite the eq. (refmoment) as follows

$$\int_{\Omega(t)} \left(\frac{\partial(\rho u_i)}{\partial t} + \mathbf{d}\mathrm{iv}(\rho v_i\mathbf{u})\right) d\upsilon = \int_{\Omega(t)} \rho\mathbf{f}d\upsilon + \int_S \mathbf{t}dS \tag{3}$$

where $\mathbf{f}$ is the extraneous force per unit mass ,$\mathbf{d}\mathrm{iv}(\rho\mathbf{u}) = \frac{\partial(\rho u_1)}{\partial x} + \frac{\partial(\rho u_2)}{\partial y} + \frac{\partial(\rho u_3)}{\partial z}$. Assuming that flow is steady (all partial derivative with respect to t are equal zero $\frac{\partial\rho u}{\partial t} = 0$), that $\mathbf{t} = -p\mathbf{n}$ and assuming also that the extraneous force per unit mass can be neglect, applying the divergence theorem one can write the momentum equation in vector form as follows

$$\boxed{\int_S \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n})dS = -\int_S p\mathbf{n}dS} \tag{4}$$

2 Application of the linear momentum equation

Example 1. *Find the reaction force exerted on a fix vane when a jet discharging $60\ l/s$ of water at $50\ m/s$ is deflected through 135^o. At first we define the reaction force. Due to fact that the jet with the vane is surrounded by atmospheric pressure p_a we define the reaction force as*

$$\mathbf{R} = \int_{S_{wall}} (p - p_a)\mathbf{n}dS \tag{5}$$

Then we applied the vector equation (4) to the control value marked as CV.

$$\int_{S_{in}} \rho\mathbf{u}(\mathbf{u}\cdot\mathrm{n})dS + \int_{S_{out}} \rho\mathbf{u}(\mathbf{u}\cdot\mathrm{n})dS = -\int_{S_{in}} p_a\mathbf{n}dS - \int_{S_{wall}} p\mathbf{n}dS - \int_{S_{out}} p_a\mathbf{n}dS - \int_{S_{free}} p_a\mathbf{n}dS \tag{6}$$

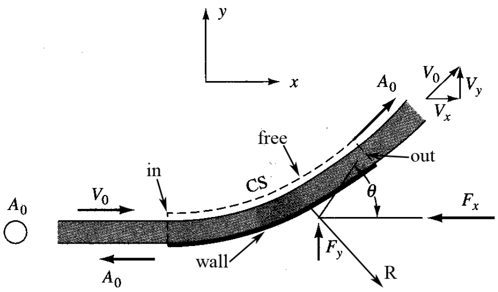

Figure 1: Free jet impinging on smooth, fixed vane

Now one can proved using the Gauss theorem (see below) that

$$\int_{S_{in}\cup S_{wall}\cup S_{out}\cup S_{free}} p_a \mathbf{n} dS = 0 \tag{7}$$

Using Eq. (7) and definition of reaction (16) equation (6) one can write

$$\int_{S_{in}} \mathbf{u}(\mathbf{u}\cdot\mathbf{n}) + \int_{S_{out}} \mathbf{u}(\mathbf{u}\cdot\mathbf{n}) + \mathbf{R} = 0 \tag{8}$$

Now, using the mean value theorem from Calculus we can write

$$\int_{S_{in}} \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})dS = -\mathbf{u}_{in}\rho q_v, \qquad \textit{where} \qquad q_v = \int(\mathbf{u}\cdot\mathbf{n})dS \tag{9}$$

And also

$$\int_{S_{out}} \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})dS = \mathbf{u}_{out}\rho q_v \tag{10}$$

So the vector equation one can write

$$-\mathbf{v}_{in}\rho q_v + \mathbf{v}_{out}\rho q_v + \mathbf{R} = 0 \tag{11}$$

The sign minus in the first term results from the fact that the normal vector $\mathbf{n}$ *is always in opposite direction to the velocity* $\mathbf{u}_{in}$*. Now, if we want to calculate the component of the reaction* $\mathbf{R} = (R_x, R_y)$ *we must project the forces on the x-axis and y-axis.*

$$R_x = V_0\rho q_v - V_0\rho q_v\cos\Theta = V_0\rho q_v(1-\cos\Theta), \qquad R_y = -V_0\rho\sin\Theta$$

Example 2. *Fluid issues from a long slot and strikes against a smooth inclined flat plate (see Figure). Determine the division of the flow and the force exerted on the plate, neglecting losses due to impact. We write the vector equation (11) for control value marked by dashed line*

$$-\mathbf{u}_{in}\rho q_v + \mathbf{u}_{out}\rho q_v + \mathbf{R} = 0 \tag{12}$$

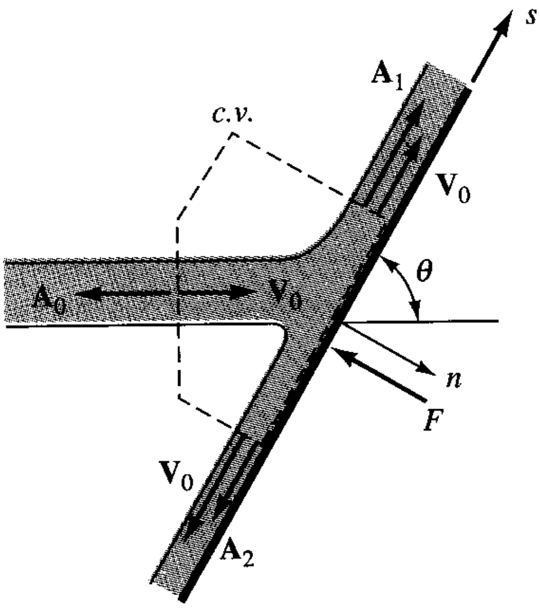

Figure 2: Free jet impinging on an inclined fixed plane surface

Projecting above equation on axis along the plate and due to fact that we assumed that the fluid is inviced, no force is exerted in the fluid by the plate in this direction, we have

$$-V_0\rho q_v cos(\Theta)+\rho V_1 q_{v_1}-\rho V_2 q_{v_2}=0$$

By continuity equation

$$q_v = q_{v_1}+q_{v_2} \tag{13}$$

Due to fact that no forces that would be able to change the velocity of the fluid particle we assumed also

$$V_0 = V_1 = V_2 \tag{14}$$

From equations (12), (13), (14) we obtain

$$q_{v_1}=\frac{q_v}{2}(1+\cos\Theta), \qquad q_{v_2}=\frac{q_v}{2}(1-\cos\Theta) \tag{15}$$

The force $F=-R_n$ exerted on the plate must be normal to it. For momentum equation normal to the plate

$$F = R_n = \rho V_0 q_v \sin\Theta$$

Example 3. *A 10–cm fire hose with 3–cm nozzle discharges 1.5 m^3/min to atmosphere. Assuming frictionless flow, find the force F_B exerted by the flange bolts to hold the nozzle on the hose.*

Solution. *We define hydrodynamic force as*

$$\mathbf{R} = \int_{S_{wall}} (p - p_a)\mathbf{n}dS \tag{16}$$

We write the momentum equation for the control volume (CV) (see figure)

$$\int_{S_1} \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})dS + \int_{S_2} \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})dS = -\int_{S_{wall}} p\mathbf{n}dS - \int_{S_1} p\mathbf{n}dS - \int_{S_2} p\mathbf{n}dS \tag{17}$$

From equation (7) we have

$$\int_{S_1\cup S_2\cup S_{wall}} p_a\mathbf{n}dS = 0 \tag{18}$$

Adding equation (18) to the left side of equation (17) we obtain

$$\int_{S_1} \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})dS + \int_{S_2} \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})dS = -\int_{S_{wall}} (p-p_a)\mathbf{n}dS - \int_{S_1} (p-p_a)\mathbf{n}dS - \int_{S_2} (p_a-p_a)\mathbf{n}dS \tag{19}$$

But the $(p - p_a)$ *is a gage pressure at given cross section. So we can write equation (19) as*

$$-p_{gage_1}A_1 - v_1\rho q_v + v_2\rho q_v + R_x = 0 \tag{20}$$

Projecting the momentum equation (20) on x-axis we obtain reaction force:

$$R_x = u_1\rho q_v - u_2\rho q_v + p_{gage_1}A_1 \tag{21}$$

The velocities u_1 *and* u_2 *we obtain from continuity equation* $q_v = u_1A_1 = u_2A_2$, $u_1 = \frac{q_v}{A_1} = 0.025/((pi/4)0.03^2) = 35.4$, $u_2 = 3.2$. *The gage pressure we find by applying the Bernoulli law for cross section*1–2.

$$p_1 + \frac{\rho v_1^2}{2} = p_b + \frac{\rho v_2^2}{2}$$

We know that $p_2 = p_a$ *and* $p_{gage_1} = p_1 - p_a = 1/2\rho(u_2^2 - u_1^2)$. *So*

$$p_{gage_1} = \frac{1}{2}\left[1000(35.4^2 - 3.2^2)\right] = 620\ 000\ Pa$$

$A_1 = \frac{\pi D_1^2}{4} = 0.00785\ m^2$. *And at last*

$$R_x = p_{gage_1}A_1 - \rho q_v(u_2 - u_1) = 4067\ N$$

3 Elementary facts from Vector Calculus

3.1 Divergence

Divergence operator in fluid mechanics play important role in analysis of flow so it is worth to spend some time to better understanding this operator.At first let start to define the divergence in one dimension. Imagine a current of water flowing along the real line $\mathbb{R}$. For each point $x \in \mathbb{R}$,

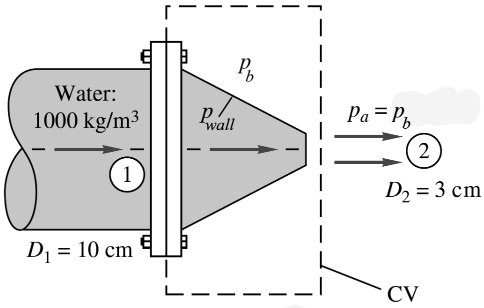

Figure 3: The fire hose with a nuzzle

let $u(x)$ describe the rate at which water is flowing past this point. Now, in places where the water slows down, we expect the derivative $u'(x)$ to be negative. We also expect the water to accumulate at such locations (because water is entering the region more quickly than it leaves). In places where the water speeds up, we expect the derivative $u'(x)$ to be positive, and we expect the water to be depleted at such locations (because water is leaving the region more quickly than it arrives).Thus, we define the **divergence** of the flow to be the rate at which water is being depleted

$$\boxed{\mathbf{d}\text{iv } u(x) = u'(x)} \tag{22}$$

For two-dimensional case to each point (x,y) we attach the vector $(u_1(x,y), u_2(x,y))$, $u(x,y) = u_1(x,y)\mathbf{i} + u_2(x,y)\mathbf{j}$. One can think as a two–dimensional current as a superposition of a horizontal current u_1 and vertical current u_2. If the horizontal current is speed–up, we expect it to deplete the fluid at this location. If it is slows-down, we expect it to deposit fluid at this location. The divergence of the two–dimensional current is thus just the sum of the divergence of its one-dimensional components:

$$\boxed{\mathbf{d}\text{iv } \mathbf{u}(x,y) = \frac{\partial u_1(x,y)}{\partial x} + \frac{\partial u_2(x,y)}{\partial y}} \tag{23}$$

Analogically we can define the divergence in three dimension

$$\boxed{\mathbf{d}\text{iv } \mathbf{u}(x,y,z) = \frac{\partial u_1(x,y,z)}{\partial x} + \frac{\partial u_2(x,y,z)}{\partial y} + \frac{\partial u_3(x,y,z)}{\partial z}} \tag{24}$$

The divergence measures the rate at which vector field $\mathbf{u} = (u_1, u_2, u_3)$ is "diverging" or "converging" near (x,y,z). Notices that divergence $\mathbf{d}$iv $\mathbf{u}$ is a scalar function. We say that vector field u is incompressible when $\mathbf{d}\text{iv } \mathbf{u} = 0$

3.2 The flux of vector field

Consider a fluid flow field

$$\mathbf{u}(x,y,z) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \tag{25}$$

and a surface S in Space. Call one side of S positive and the other side negative, and at each point of S let $\mathbf{n}$ be the unit vector on the positive side of S. The surface integral

$$\int_S \mathbf{u}\cdot\mathbf{n}dS \tag{26}$$

will be the flux, or net rate of fluid flow across the surface S from the negative to the positive side. Notice that uder the integral we have the scalar (dot) product. It is define as $\mathbf{u}\cdot\mathbf{n} = \sum_{i=1}^{3} u_i\, n_i$

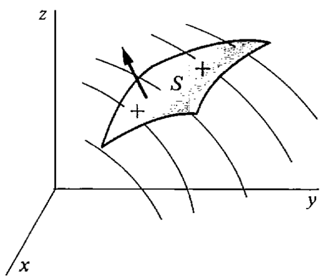

Figure 4: Oriented surface with unit normal vector

3.3 Gauss' theorem (the divergence theorem)

Gauss's theorem corresponds to a quite simple and rather intuitive idea: the integral of a derivative equals the net value of the function (whose derivative is being integrated) over the boundary of the domain of integration. In one space dimension this is precisely the *fundamental theorem of calculus*:

$$\int_a^b F'(x)dx = F(b) - F(a) \tag{27}$$

This basic idea contained in (27) generalizes to two dimensions in the form of Gauss's theorem which is also known as the **divergence theorem**.

Theorem 1. *(Gauss, or divergence). For any smooth vector filed* $\mathbf{F}$ *over a region* $R \in \mathbb{R}$ *with a smooth boundary S*

$$\int_R \mathbf{d}\mathrm{iv}\mathbf{F} = \int_S \mathbf{F}\cdot\mathbf{n}dA \tag{28}$$

In figure (5) is shown the geometry of surface integration, showing the vector $\mathbf{F}$ over a differential element of surface dA. Also the outward unit vector normal $\mathbf{n}$ and projection of $\mathbf{F}$ onto $\mathbf{n}$, $\mathbf{F}\cdot\mathbf{n}$. The integral over S is just the sum of these projections multiplied by their corresponding differential areas, in the limit area of the patches approaching zero.

4 Problems

Fluid with constant velocity U_∞ and density ρ flows past along the plate$(u) = U_\infty\mathbf{i}$. The pressure is assumed uniform, and so it has no net force on the plate. The only effect of the plate is due to

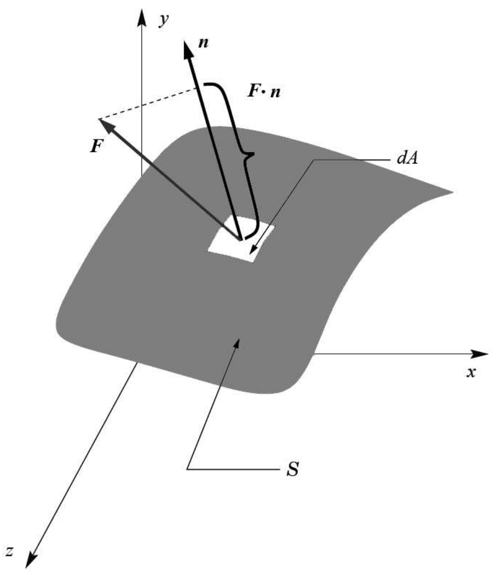

Figure 5: Integration of a vector field over a surface

boundary shear due to viscosity of the fluid. The no–slip condition at the wall brings the fluid near the boundary to a halt, and these slowly moving particles retard their neighbors above, so that at the end of the plate there is a significant retarded boundary layer of thickness $y = \delta$. Apply the momentum equation (2) and find the drag force D, ($D = -\int_{S_{wall}} \mathbf{t} dS, S = bL$)(Answer:

$$D = \rho b \int_0^{\delta} u(U_\infty - u) dy \,|_{x=L}$$

. During this course I will be used the following books:

References

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[3] E. J. Shaughnessy, Jr.,I. M. Katz and J. P. Schaffer, 2005. *Introduction to Fluid Mechnaics*, Oxford University Press.

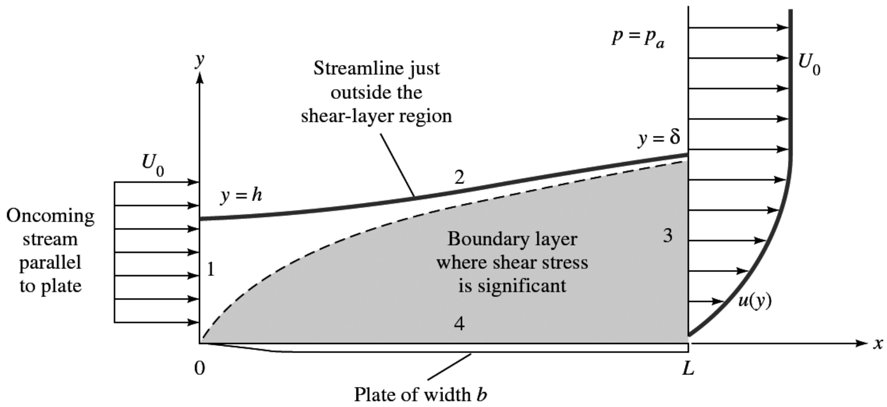

Figure 6: Analysis of the drag force on a flat plate due to boundary share by applying the linear momentum principle.