External Flows. Boundary Layer concepts

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1 Introduction

External flows past objects encompass an extremely wide variety of fluid mechanics phenomena. Clearly the character of the flow field is a function of the shape of the body. For a given-shaped object,the characteristics of the flow depend very strongly on various parameters such as size,orientation,speed,and fluid properties. According to dimensional analysis arguments,the character of the flow should depend on the various dimensionless parameters involved. For typical external flows the most important of these parameters are the Reynolds number$Re = UL/\nu$, where L– is characteristic dimension of the body. For many high-Reynolds-number flows the flow field may be divided into two region

1. a viscous boundary layer adjacent to the surface of the vehicle
2. the essentially inviscid flow outside the boundary layer

W know that fluids adhere the solid walls and they take the solid wall velocity. When the wall does not move also the velocity of fluid on the wall is zero. In region near the wall the velocity of fluid particles increases from a value of zero at the wall to the value that corresponds to the external ”frictionless” flow outside the boundary layer (see figure).

2 Boundary layer concepts

The concept of the boundary layer was developed by Prandtl in 1904. It provides an important link between ideal fluid flow and real-fluid flow.

> Fluids having relatively small viscosity , the effect of internal friction in a fluid is appreciable only in a narrow region surrounding the fluid boundaries.

Since the fluid at the boundaries has zero velocity, there is a steep velocity gradient from the boundary into the flow. This velocity gradient in a real fluid sets up shear forces near the boundary

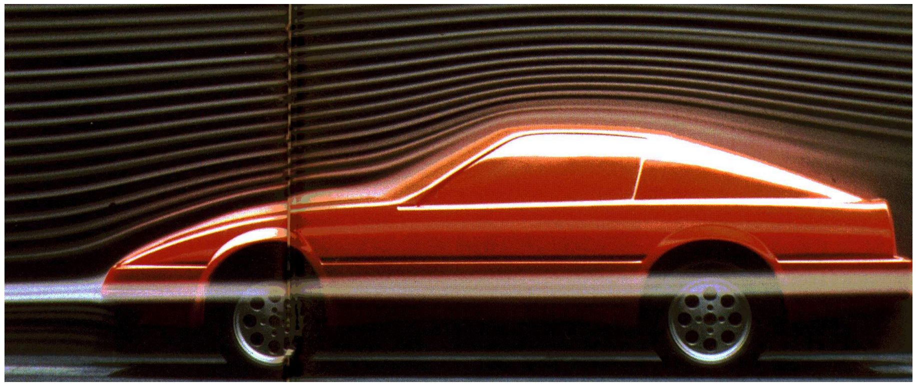

Figure 1: Visualization of the flow around the car. It is visible the thin layer along the body cause by viscosity of the fluid. The flow outside the narrow regin near the solid boundary can be considered as ideal (ivicied).

that reduce the flow speed to that of the boundary.That fluid layer which has had its velocity affected by the boundary shear is called ***the boundary layer***.
For smooth upstream boundaries the boundary layer starts out as a ***laminar boundary*** layer in which the fluid particles move in smooth layers. As the laminar boundary layer increases in thickness, it becomes unstable and finally transforms into a ***turbulent boundary layer*** in which the fluid particles move in haphazard paths. When the boundary layer has become turbulent, there is still a very thon layer next to the boundary layer that has laminar motion. It is called the ***laminar sublayer***.
Various definitions of boundary–layer thickness δ have been suggested. The most basic definition

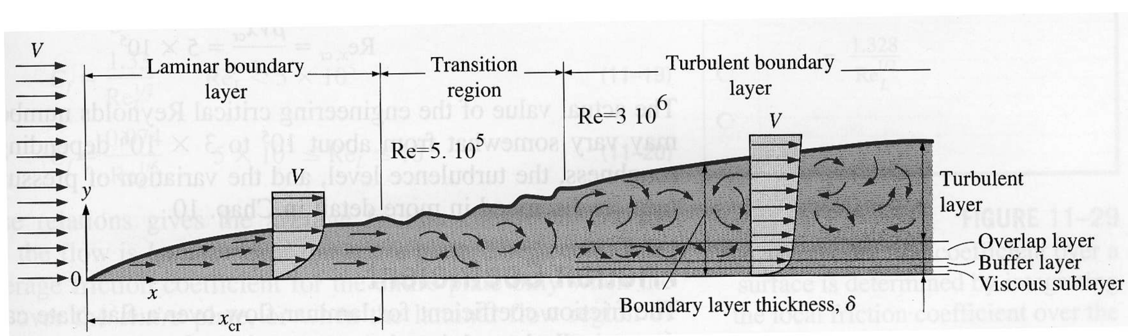

Figure 2: The development of the boundary layer for flow over a flat plate, and the different flow regimes. The vertical scale has been greatly exaggerated and horizontal scale has been shortened.

refers to the displacement of the main flow due to slowing down od particles in the boundary zone.

This thickness δ_1^*,called ***the displacement thickness*** is expressed by

$$\boxed{U\delta_1{}^* = \int_0^\delta (U-u)dy} \quad (1)$$

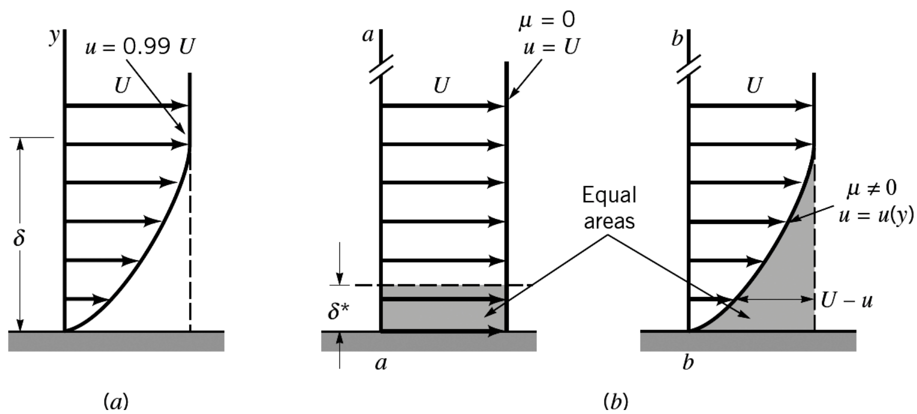

Figure 3: Definition of boundary layer thickness:(a) standard boundary layer($u = 99\%U$),(b) boundary layer displacement thickness .

The boundary layer thickness is defined also as that distance from the plate at which the fluid velocity is within some arbitrary value of the upstream velocity. Typically, as indicated in figure (**??**a), $\delta = y$ where $u = 0.99U$.
Another boundary layer characterstic, called as ***the boundary layer momentum thickness***, Θ as

$$\boxed{\Theta = \int_0^\delta \frac{u}{U}(1-\frac{u}{U})dy} \quad (2)$$

All three boundary layer thickness definition δ, δ_1^*, Θ are use in boundary layer analysis.

3 Scaling analysis

Prandtl obtained the simplified equation of fluid motion inside the boundary layer by scaling analysis called a r*relative order of magnitude analysis*. Let as recall the steady equation of motion for longitudinal component of velocity

$$u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (3)$$

The left terms of the eq. (**??**) is called as advective term of acceleration. The term proportional to the viscosity represent viscous forces. At first Prandtl's boundary layer theory is a applicable if $\delta \ll L$, it is the thickness of the boundary layer is much smaller that the then streamwise

(longitudinal) length of body.
Let a characteristic magnitude of u in the flow field be U. Let L be the streamwise distance over which u changes appreciably (from 0 to U). A measure of $\frac{\partial u}{\partial x}$ is therefore $\frac{U}{L}$, so that the advective term $u\frac{partialu}{\partial x}$ may be estimated

$$u\frac{\partial u}{\partial x} \sim \frac{U^2}{L} \tag{4}$$

where $\sim$ is to be interpreted as "of order". We can regarded the term $\frac{U^2}{L}$ as a measure of the inertial forces. A measure of the viscus term in eq. (**??**) is

$$\nu\frac{\partial^2 u}{\partial y^2} \sim \frac{\nu U}{\delta^2} \tag{5}$$

The term ($\frac{\partial^2 u}{\partial x^2} \sim \frac{U}{L^2}$ is much smaller than term ($\frac{\partial^2 u}{\partial y^2} \sim \frac{U}{L^2}$ may be drops from the equations. Prandtl assumed that within the boundary layer, the viscous forces and inertial forces are the same order. It means that

$$\frac{\nu U}{\delta^2} : \frac{U^2}{L} = \frac{\nu}{LU}\left(\frac{L}{\delta}\right)^2 =\sim 1 \tag{6}$$

Recognizing that $UL/\nu = Re_L$,we see immediately that

$$\delta \sim \frac{L}{\sqrt{Re_L}} \tag{7}$$

The coefficient of the proportionality, that correspond to the thickness of boundary layer according

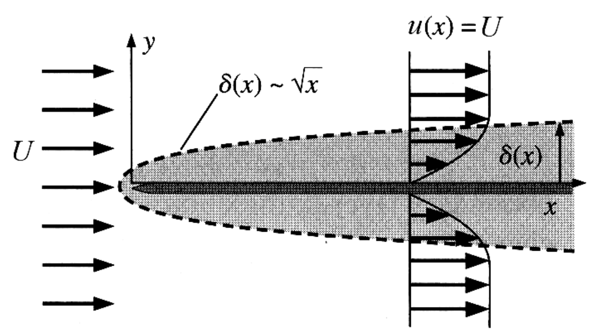

Figure 4: An order–of–magnitude analysis of the laminar boundary layer equations along a flat plate revels that δ grows like $\sqrt{x}$

to the definition of δ as $u = 0.99U$ in eq. (**??**) is commonly taken as equal to 5. The thickness of the boundary layer along the flat plate depend on x and can be calculated form

$$\boxed{\delta(x) = \frac{5x}{\sqrt{Re_x}}, \qquad Re_x = \frac{Ux}{\nu}} \tag{8}$$

The set of equations of the motion for a steady, incompressible laminar boundary layer in xy–plane without significant gravitational effects are

$$u\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial x}+v\frac{\partial^2 u}{\partial y^2} \tag{9}$$

$$0=-\frac{\partial p}{\partial y} \tag{10}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{11}$$

Due to fact that in boundary layer $v \ll u$ y–components equation of momentum reduced to the (**??**). Equation (**??**) says that the pressure is approximately uniform (constant) across the boundary layer. The pressure at the surface is therefore equal to that at the edge of the boundary layer we can apply the Bernoulli equation to the outer flow region. Differentiating with respect to x the Bernoulli equation we obtain

$$\frac{p}{\rho}+\frac{1}{2}U^2=const \qquad \rightarrow \qquad \frac{1}{\rho}\frac{\partial p}{\partial x}=-U\frac{dU}{dx}$$

The equations (**??**) and (**??**) are used to determine u and v in boundary layer. The boundary layer conditions are

$$u(x,0)=0 \tag{12}$$

$$v(x,0)=0 \tag{13}$$

$$u(x_0,y)=u_{in}(y) \tag{14}$$

4 Momentum Equation Applied to the Boundary Layer

In order to calculate boundary layers approximately, we often use methods where the equations of motion are not satisfied everywhere in the field but only in integral means across the thickness of the boundary layer. The starting point for these integral methods is usually the momentum equation which can be derived by applying the continuity equation and the balance of momentum in its integral form. Let us applied the balance of momentum to the flow over the flat plate. The control volume was chosen as it is shown in figure (**??**). Pay attention that due to action of the boundary layer the streamline 2 is displaced from the solid wall. There is no mass flow through the streamline 2. The equation of the linear momentum is

$$\boxed{\int_S \rho\mathbf{u}(\mathbf{u}\cdot n)dS=\int_S \mathbf{t}dS=-F_D} \tag{15}$$

The pressure is assumed uniform, and so it has no net force on the plate. Evaluating the integral in on left side of the eq. (**??**) on obtain

$$-F_D=\rho\int_1 u(\mathbf{u}\cdot\mathbf{n}dS+\rho\int_3 u(\mathbf{u}\cdot\mathbf{n}dS=\rho\int_0^h U_0(-U_0)b\,dy+\rho\int_0^\delta uub\,dy=-\rho U_0^2 bh+\rho b\int_0^\delta\,dy \tag{16}$$

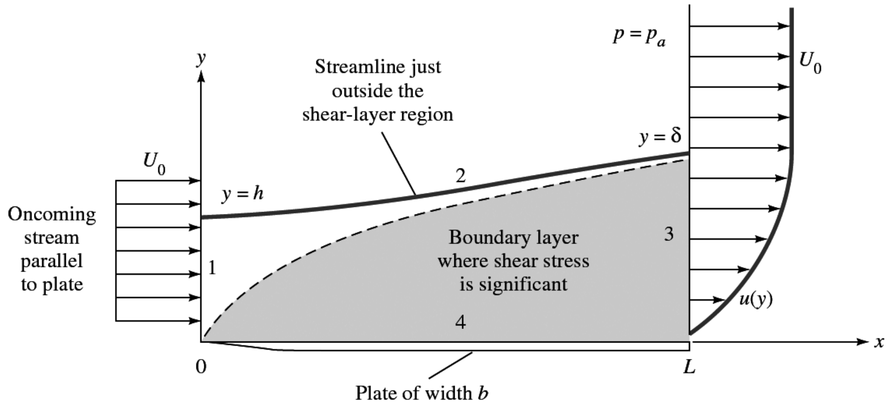

Figure 5: Analysis of the drag force on a flat plate due to boundary share by applying the linear momentum principle.

or

$$F_D = \rho U_0^2 bh - \rho b \int_0^{\delta} \; dy \tag{17}$$

where b is a width of the plate. Now using the integral form of continuity equation (conservation of mass = the flow rate through section (1) must equal that through section (2)) we obtain

$$\rho b \int_0^h vektu \cdot \mathbf{n} \; dy = bU\rho h = \rho \int_0^{\delta} u \; dy \tag{18}$$

Introduce the value of h to (**??**) we obtain

$$F_D = \rho b \int_0^{\delta} u(U-u) \; dy|_{x=L} \tag{19}$$

The development of Eq.(**??**and its use was first derived by Theodore von Karman in 1921. By comparing Eqs. (**??**) and (**??**) we see that the drag can be written in terms of the momentum thickness Θ, as

$$F_D = \rho b U^2 \Theta \tag{20}$$

Momentum thickness Θ measure of total plate drag. But the integral wall shear stress along the plate gives also the drag force

$$F_D(x) = b \int_0^x \tau_w(x) \; dx, \quad \text{and} \quad \frac{dF_D}{dx} = b\tau_w \tag{21}$$

Meanwhile, the derivative of eq. (**??**), with U=contstant, is

$$\frac{dF_D}{dx} = \rho b U^2 \frac{d\Theta}{dx} = b\tau_w \tag{22}$$

Equation

$$\boxed{\rho U^2 \frac{d\Theta}{dx} = \tau} \tag{23}$$

ix called the ***momentum-integral relation*** for the flat–plate boundary layer flow.
The equation (**??** one can also write in non– dimensional form

$$\boxed{\frac{d\Theta}{dx} = \frac{\tau}{\rho U^2} = \frac{1}{2} C_f(x)} \tag{24}$$

where $\tau_w/\rho U^2/2$ is defined as a skin friction coefficient, C_f. The point is to use eqn. (**??**) to determine u/U when we do not have an exact solution. To do this, we guess the solution in the form $u/U = f(y/\delta)$. This guess is made in such a way that it will fit the following four things that are true of the velocity profile (boundary values on the solid wall, $y = 0$, and on the edge of boudary layer, $y = \delta$):

$$u = 0 \qquad \text{for} \quad y = 0 \quad (\text{on the solid boundary}) \tag{25}$$

$$u = U \qquad \text{at} \quad y = \delta \quad (\text{on the edge of boundary layer}) \tag{26}$$

$$\frac{du}{dy} = 0 \qquad \text{at} \quad y = \delta \tag{27}$$

$$\frac{d^2u}{dy^2} = 0 \qquad \text{at} \qquad y = 0 \tag{28}$$

If $f(y/\delta)$ is written as a polynomial with four constants a, b ,c and d

$$\frac{u}{U} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2 + d\left(\frac{y}{\delta}\right)^3 \tag{29}$$

the four bove condtion about teh velocity porfile give

- $0 = a$, which eliminate a immiediatly
- $1 = b + c + d$
- $0 = b + 2c + 3d$
- $0 = 2c$

Solving the middke two above equations for b and d we obain $d = -1/2$ and $b = 3/2$, so

$$\frac{u}{U} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \tag{30}$$

This makes possible to estimate both momentum thicness and wall shear

$$\Theta = \int_0^\delta \left(\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right)\left(1 - \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right) dy = \delta\frac{39}{280} \tag{31}$$

$$\tau_w = \mu\frac{\partial u}{\partial y}|_{y=0} = \frac{3}{2}\frac{\mu U}{\delta} \tag{32}$$

By substituting (**??**) into (**??**) and rearranging we obtain

$$\delta d\delta = \frac{140}{13}\frac{\nu}{U}dx \tag{33}$$

where $\nu = \mu/\rho$. We can integrate from 0 into x, assuming that $\delta = 0$ at $x = 0$

$$\delta^2 = \frac{280}{13}\frac{\nu x}{U} \tag{34}$$

or

$$\frac{\delta}{x} = \frac{4.64}{\sqrt{Re_x}} \tag{35}$$

This is the desired thickness estimate. It is accurate, being 5.6% smaller than the known exact solution for laminar flat–plate flow ($\delta/x = 5/\sqrt{Re_x}$)
we can also obtain a shear–stress, skin–friction coefficient

$$C_f = \frac{2\tau_2}{\rho U} = \frac{3}{4.64}\frac{1}{\sqrt{Re_x}} = \frac{0.6466}{\sqrt{Re_x}}, \qquad Re_x = \frac{Ux}{\nu} \tag{36}$$

Exact laminar–plate–solution is $C_f = 0.664/Re^{1/2}$.

During this course I will be used the following books:

References

[1] F. M. White, 1999. *Fluid Mechanics*, McGraw-Hill.

[2] B. R. Munson, D.F Young and T. H. Okiisshi, 1998. *Fundamentals of Fluid Mechanics*, John Wiley and Sons, Inc. .