

# Viscous flow in pipe

Henryk Kudela

## Contents

<b>1</b>	<b>Laminar or turbulent flow</b>	<b>1</b>
<b>2</b>	<b>Balance of Momentum - Navier-Stokes Equation</b>	<b>2</b>
<b>3</b>	<b>Laminar flow in pipe</b>	<b>2</b>
3.1	Friction factor for laminar flow . . . . .	5

## 1 Laminar or turbulent flow

The flow of a fluid in a pipe may be laminar flow or it may be turbulent flow. Osborne Reynolds (1842-1912), a British scientist and mathematician, was the first to distinguish the difference between these two classifications of flow by using a simple apparatus as shown in Fig.1.

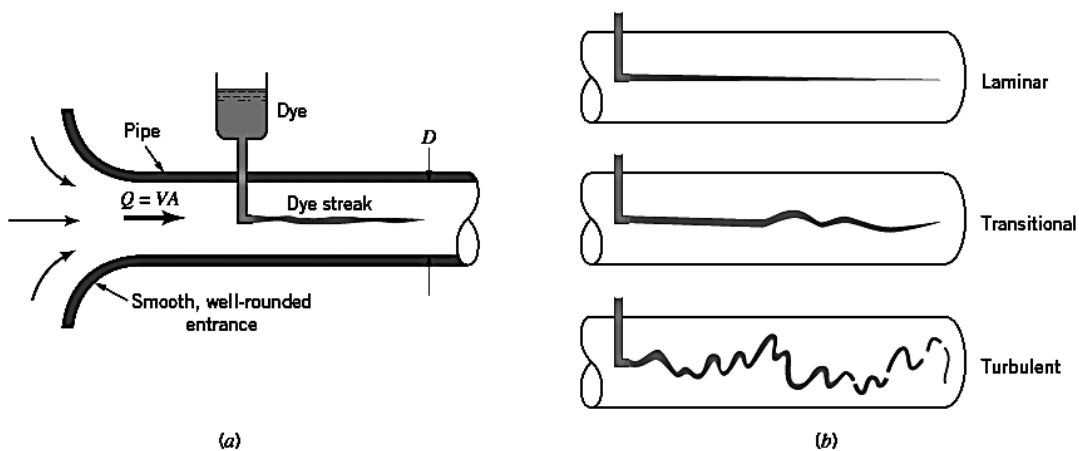


Figure 1: (a) Reynolds' experiment using water in a pipe to study transition to turbulence, (b) Typical dye streaks

If water runs through a pipe of diameter  $D$  with an average velocity  $V$ , the following characteristics are observed by injecting neutrally buoyant dye as shown. For small enough flow rates the dye streak (a streakline) will remain as a well-defined line as it flows along, with only slight blurring due to molecular diffusion of the dye into the surrounding water. For a somewhat larger intermediate flowrate the dye streak fluctuates in time and space, and intermittent bursts of irregular behavior appear along the streak. On the other hand, for large enough flow rates the dye streak almost immediately becomes blurred and spreads across the entire pipe in a random fashion. These three characteristics, denoted as laminar, transitional, and turbulent flow, respectively, are illustrated in Fig. 1b.

For pipe flow the most important dimensionless parameter is the Reynolds number,  $Re = VD/\nu$  the ratio of the inertia to viscous effects in the flow. Hence, the term flowrate should be replaced by Reynolds number, where  $V$  is the average velocity in the pipe. That is, the flow in a pipe is laminar, transitional, or turbulent provided the Reynolds number is small enough, intermediate, or large enough. It is not only the fluid velocity that determines the character of the flow its density, viscosity, and the pipe size are of equal importance. These parameters combine to produce the Reynolds number.

The distinction between laminar and turbulent pipe flow and its dependence on an appropriate dimensionless quantity was first pointed out by Osborne Reynolds in 1883. The Reynolds number ranges for which laminar, transitional, or turbulent pipe flows are obtained cannot be precisely given. The actual transition from laminar to turbulent flow may take place at various Reynolds numbers, depending on how much the flow is disturbed by vibrations of the pipe, roughness of the entrance region, and the like. For general engineering purposes (i.e., without undue precautions to eliminate such disturbances), the following values are appropriate: The flow in a round pipe is laminar if the Reynolds number is less than approximately **2100**. The flow in a round pipe is turbulent if the Reynolds number is greater than approximately **4000**. For Reynolds numbers between these two limits, the flow may switch between laminar and turbulent conditions in an apparently random fashion (transitional flow).

## 2 Balance of Momentum - Navier-Stokes Equation

Differential form of the **balance principle of momentum** (see lecture n3-equation motion) for viscous fluid take the differential form which are called Navier-Stokes equations (N-S). Further we will treat the fluid as a incompressible. The vector form of N-S equations are

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{v} \quad (1)$$

$$\nabla \cdot \mathbf{v} = 0 \quad (2)$$

For viscous flow relation between the the vector of tension  $\mathbf{t}(x,t)$  is not like for **ideal** (inviscid) fluid,  $\mathbf{t} = -p \cdot \mathbf{n}$ , but depend also from derivatives  $\frac{\partial v_i}{\partial v_j}$ . The consequences of that fact are the Navier-Stokes equations (1).

### 3 Laminar flow in pipe

We will be regarded the flow in long, straight, constant diameter sections of a pipe as a fully developed laminar flow. The gravitational effects (mass forces) will be neglect. The velocity profile is the same at any cross section of the pipe. Although most flows are turbulent rather than laminar, and many pipes are not long enough to allow the attainment of fully developed flow, a theoretical treatment and full understanding of fully developed laminar flow is of considerable importance. First, it represents one of the few theoretical viscous analysis that can be carried out exactly (within the framework of quite general assumptions) without using other ad hoc assumptions or approximations. An understanding of the method of analysis and the results obtained provides a foundation from which to carry out more complicated analysis. Second, there are many practical situations involving the use of fully developed laminar pipe flow.

We isolate the cylinder of fluid as is shown in Fig. 2 and apply Newtons second law,  $\frac{d(mv_x)}{dt} = F_x$ . In this case even though the fluid is moving, it is not accelerating, so that  $\frac{d(mv_x)}{dt} = 0$ . Thus, fully developed horizontal pipe flow is merely a balance between pressure and viscous forces—the pressure difference acting on the end of the cylinder of area  $\pi r^2$  and the shear stress acting on the lateral surface of the cylinder of area  $2\pi r l$ . This force balance can be written as

$$p_1 \pi r^2 - (p_1 - \Delta p) \pi r^2 - 2\pi r l \tau = 0$$

which can be simplified to give

$$\frac{\Delta p}{l} = \frac{2\tau}{r} \quad (3)$$

Equation (3) represents the basic balance in forces needed to drive each fluid particle along the pipe with constant velocity. Since neither  $\Delta p$  are functions of the radial coordinate,  $r$ , it follows that  $2\tau/r$  must also be independent of  $r$ . That is,  $\tau = Cr$  where  $C$  is a constant. At  $r = 0$  (the centerline of the pipe) there is no shear stress ( $\tau = 0$ ). At  $r = D/2$  (the pipe wall) the shear stress is a maximum, denoted  $\tau_w$  the *wall shear stress*. Hence,  $C = 2\tau_w/D$  and the shear stress distribution throughout the pipe is a linear function of the radial coordinate

$$\tau = \frac{2\tau_w r}{D} \quad (4)$$

The linear dependence of  $\tau$  on  $r$  is a result of the pressure force being proportional to  $r^2$  (the pressure acts on the end of the fluid cylinder; area= $\pi r^2$ ) and the shear force being proportional to  $r$  (the shear stress acts on the lateral sides of the cylinder; area= $2\pi r l$ ). If the viscosity were zero there would be no shear stress, and the pressure would be constant throughout the horizontal pipe. The pressure drop and wall shear stress are related by

$$\Delta p = \frac{4l\tau_w}{D} \quad (5)$$

A small shear stress can produce a large pressure difference if the pipe is relatively long ( $l/D \gg 1$ ).

Although we are discussing laminar flow, a closer consideration of the assumptions involved in the derivation of (3),(4) and (5) reveals that these equations are valid for both laminar and

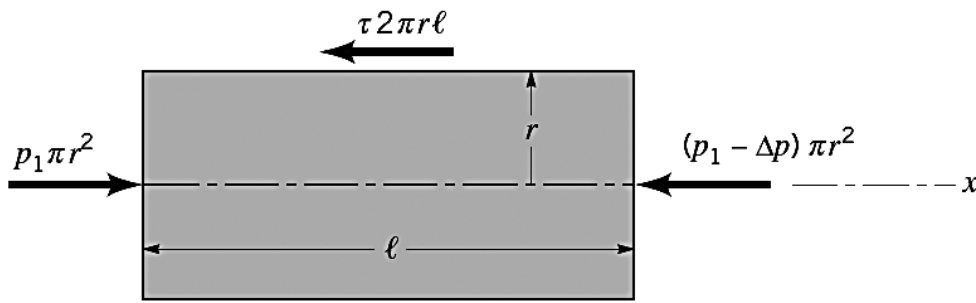
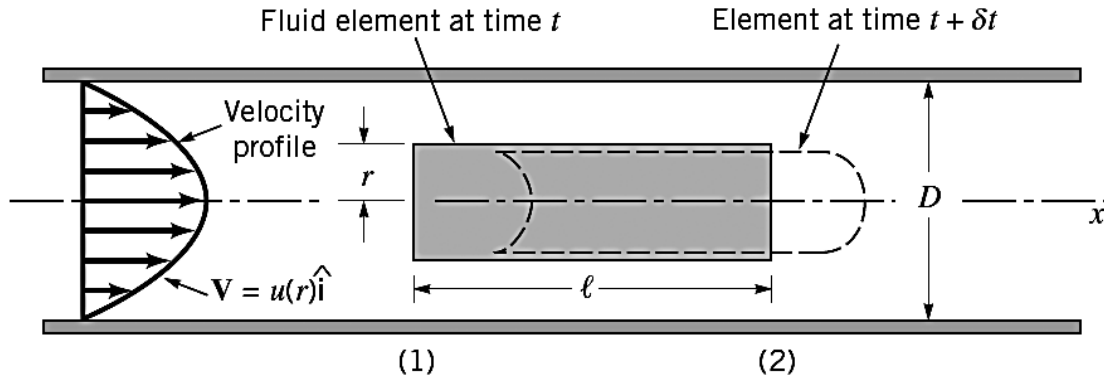


Figure 2: Motion of cylindrical fluid element within a pipe

turbulent flow. To carry the analysis further we must prescribe how the shear stress is related to the velocity. This is the critical step that separates the analysis of laminar from that of turbulent flow—from being able to solve for the laminar flow properties and not being able to solve for the turbulent flow properties without additional ad hoc assumptions. The shear stress dependence for turbulent flow is very complex. However, for laminar flow of a **Newtonian fluid**, the shear stress is simply proportional to the velocity gradient,  $\tau = \mu du/dy$ . In the notation associated with our pipe flow, this becomes

$$\tau = -\mu \frac{du}{dr} \quad (6)$$

The negative sign is included to give  $\tau > 0$  with  $du/dr < 0$  (the velocity decreases from the pipe centerline to the pipe wall). Equations (3) and (6) represent the two governing laws for fully developed laminar flow of a Newtonian fluid within a horizontal pipe. The one is Newton's second law of motion and the other is the definition of a Newtonian fluid. By combining these two equations we obtain

$$\frac{du}{dr} = -\left(\frac{\Delta p}{2\mu l}\right) \quad (7)$$

which can be integrated to give the velocity profile:

$$u = - \left( \frac{\Delta p}{4\mu l} \right) r^2 + C_1 \quad (8)$$

where  $C_1$  is a constant. Because the fluid is viscous it **sticks** to the pipe wall so that  $u = 0$  at  $r = D/2$ . Thus,  $C_1 = 4R^2(\Delta p/16\mu l)$ . Hence, the velocity profile can be written as

$$u(r) = \left( \frac{\Delta p R^2}{4\mu l} \right) \left[ 1 - \left( \frac{r}{R} \right)^2 \right] = V_{max} \left[ 1 - \left( \frac{r}{R} \right)^2 \right] \quad (9)$$

where  $V_{max} = \left( \frac{\Delta p R^2}{4\mu l} \right)$  is the centerline velocity.

The volume flowrate through the pipe can be obtained by integrating the velocity profile across the pipe. Since the flow is axisymmetric about the centerline, the velocity is constant on small area elements consisting of rings of radius  $r$  and thickness  $dr$ . Thus,

$$q_v = \int_A u \, dA = \int_0^R u(r) 2\pi r \, dr = 2\pi V_{max} \int_0^R \left[ 1 - \left( \frac{r}{R} \right)^2 \right] r \, dr = 2\pi V_{max} \left[ \frac{r^2}{2} - \frac{r^4}{4R^2} \right] \Big|_0^R \quad (10)$$

or or pressure drops

$$q_v = \pi R^2 \frac{V_{max}}{2} \quad (11)$$

By definition, the average velocity is the flowrate divided by the cross-sectional area,  $V = q_v/\pi R^2$  so that for this flow

$$V = \frac{V_{max}}{2} = \frac{\Delta p R^2}{8\mu l} \quad (12)$$

and

$$q_v = \frac{\pi R^4 \Delta p}{8\mu l} \quad (13)$$

Equation (13) is commonly referred to as **Hagen-Poiseuille's** law.

For horizontal pipe the flowrate is:

- directly proportional to the pressure drop
- inversely proportional to the viscosity
- inversely to the pipe length
- proportional to the pipe radius to the **fourth power** ( $\sim R^4$ )

Recall that all of these results are restricted to laminar flow (those with Reynolds numbers less than approximately 2100) in a horizontal pipe.

### 3.1 Friction factor for laminar flow

From Darcy-Weisbach

$$h_L = \frac{\Delta p}{\rho g} = f \frac{L}{D} \frac{v^2}{2g}$$

we have

$$f = \frac{\Delta p}{\frac{1}{2}\rho V^2} \frac{D}{L} \quad (14)$$

Equation (12) can be rearrange to obtain

$$\Delta p = \frac{8\mu V l}{R^2} = \frac{32\mu V}{D} \left( \frac{L}{D} \right) \quad (15)$$

Inserting (15) to (14) one obtain

$$\boxed{f = \frac{64}{Re}} \quad (16)$$

During this course I will be used the following books:

### References

- [1] F. M. White, 1999. *Fluid Mechanics*, McGraw-Hill.
- [2] B. R. Munson, D.F Young and T. H. Okiisshi, 1998. *Fundamentals of Fluid Mechanics*, John Wiley and Sons, Inc. .
- [3] Y. Nakayama and R.F. Boucher, 1999. *Intoduction to Fluid Mechanics*, Butterworth Heine-mann.