

# The vortex-in-cell method for the study of three-dimensional vortex structures

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**Abstract** The vortex particle method for numerical simulation of the 3D vortex structure evolution was used. Validation of the method was tested for the study of a single vortex ring by comparing the computed translation velocity with the theoretical formula and for the leap-frogging phenomenon for two rings with the same circulation.

*Our paper is clear as a bell  
On the method of vortex-in-cell;  
For vortical strings  
And leap-frogging rings  
The method works plausibly well.*

## 1. Introduction

Vorticity plays a fundamental role in all real fluid dynamic phenomena and for this reason the vortex method in the study of the fluid dynamics cannot be overestimated. In computation the “vortex particle” permits direct tracing for the evolution of the vorticity. Now it seems that the 2D vortex particle method is well grounded in that many numerical and theoretical results have been obtained (Ould Salihi & al 2000, Kudela 1999). On the other hand, the 3D vortex method must still be developed. Generally the vortex method can be divided on the direct, free grid method based on the Biot-Savart law (Leonard 1985, Knio & Ghoniem 1990, Winckelmans & Leonard 1993) and vortex-in-cell methods where a grid is used for the velocity calculation but particles are used to track the vorticity (Christiansen 1973, Zawadzki & Aref 1991, Cottet 2000). The vortex-in-cell method is much faster than the free grid vortex method. Despite the fact that vorticity is divergence free, we introduced to the computation a vector particle that carries the “mass” of the vorticity. We are going to build a 3D program for the simulation of the viscous fluid flow using the vortex-in-cell method. Components that must be

included in such a program should be: a Euler inviscid solution of the flow by the vortex particles and a numerical procedure that takes into account the viscosity of the fluid (Cottet 2000). Here we present our primary results that relate to the inviscid Euler equations.

## 2. Equation of motion and description of the computational algorithm

The equations that described the evolution of the vorticity in the inviscid and incompressible three-dimensional space are:

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \nabla) \vec{\omega} = (\vec{\omega} \cdot \nabla) \vec{u} \quad (1)$$

$$\nabla \cdot \vec{u} = 0 \quad (2)$$

where  $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$  is the vorticity vector and  $\vec{u} = (u_1, u_2, u_3)$  is the velocity. Now in agreement with the spirit of the vortex-in-cell method, the distribution of the vorticity is replaced by a discrete distribution of Dirac delta measures:

$$\vec{\omega}(\vec{x}) = \sum_{p=0}^N \vec{\alpha}_p \delta(\vec{x} - \vec{x}_p) \quad (3)$$

where  $\vec{\alpha}_p$  means vector particle  $\vec{\alpha}_p = (\alpha_{p1}, \alpha_{p2}, \alpha_{p3})$  at position  $\vec{x}_p = (x_{p1}, x_{p2}, x_{p3})$ . When the domain of the flow is covered by the numerical mesh  $(N_x \times N_y \times N_z)$  with equidistant spacing  $h$ , then the  $i$ -component of the vector particle  $\vec{\alpha}_p$  is:

$$\alpha_i = \int_{V_p} \omega_i(x_1, x_2, x_3) d\vec{x} \approx h^3 \omega_i(\vec{x}_p), \quad \vec{x}_p \in V_p, \quad |V_p| = h^3. \quad (4)$$

Equation (2) assures the existence of vector potential  $\vec{A}$  which relates the vorticity distribution to the velocity field:

$$\vec{u} = \nabla \times \vec{A} \quad (5)$$

where the components of vector potential  $\vec{A}$  are obtained by the solution of the Poisson equations (it was assumed that  $\nabla \cdot \vec{A} = 0$ ):

$$\Delta A_i = -\omega_i, \quad i = 1, 2, 3. \quad (6)$$

The numerical calculation goes as follows:

1) To solve equations (6) on the numerical mesh the strength of particles  $\vec{\alpha}_p$  must be redistributed on the mesh nodes  $(l, m, n)$ :

$$\omega_i(x_l, x_m, x_n) = \frac{1}{h^3} \sum_p \alpha_i(\vec{x}_p) \varphi_{lmn}(\vec{x}_p), \quad i = 1, 2, 3 \quad (7)$$

where for  $\varphi$  we used the B-spline of the first order (it is equivalent to the volume-weighted scheme and influenced only 8 nodes). After the redistribution, equations (6) were solved by the fast Poisson solver. We used the periodic boundary conditions.

2) Using (5), the velocities at the grid nodes were computed by the central difference. The particles were advanced in time employing the Runge-Kutta scheme:

$$\frac{d\vec{x}_p}{dt} = \vec{u}_p. \quad (8)$$

The velocity  $\vec{u}_p$  was computed from the grid nodes velocities by interpolation. We used the second order interpolator from the Fortran IMSL library.

3) Due to the vorticity stretching effect, in the new position the strength of the particles was updated:

$$\frac{d\alpha_i}{dt} = \alpha_1 \frac{\partial u_i(\vec{x}_p)}{\partial x_1} + \alpha_2 \frac{\partial u_i(\vec{x}_p)}{\partial x_2} + \alpha_3 \frac{\partial u_i(\vec{x}_p)}{\partial x_3}, \quad i = 1, 2, 3. \quad (9)$$

The derivatives of the velocity were interpolated from the grid nodes on the position of the particles. For the solution of (9) we used the 4-order Adams-Bashforth scheme. This completes one time step and the calculation returns to step 1.

### 3. Examples of the numerical results

At first we used our program to test the motion of a single ring with uniform vorticity inside the core. The translation velocity  $U_T$  is given by the the formula (Lim & Nickels 1995, Saffman 1993):

$$U_T = \frac{\Gamma}{4\pi R} \left\{ \log \left( \frac{8R}{\varepsilon} \right) - \frac{1}{4} + O \left( \frac{\varepsilon}{R} \right) \right\} \quad (10)$$

where  $\varepsilon$  - radius of the core,  $R$  - radius of the ring,  $\Gamma$  - circulation. For our calculation we took domain  $10 \times 10 \times 10$  and grid step  $h = 0.1$ ,

$\Gamma = 1.0$ ,  $\varepsilon = 0.3$ ,  $R = 1.5$ . The time step was used as  $\Delta t = 0.005$ . The ring was divided into 100 slices, and in each slice the vorticity was redistributed between 100 particles ( $\Gamma \cdot h \approx \sum_p \vec{\alpha}_p \cdot \vec{n}$  where the summation is made for the particles that are in volume  $(h \cdot \pi \cdot \varepsilon^2)$ ),

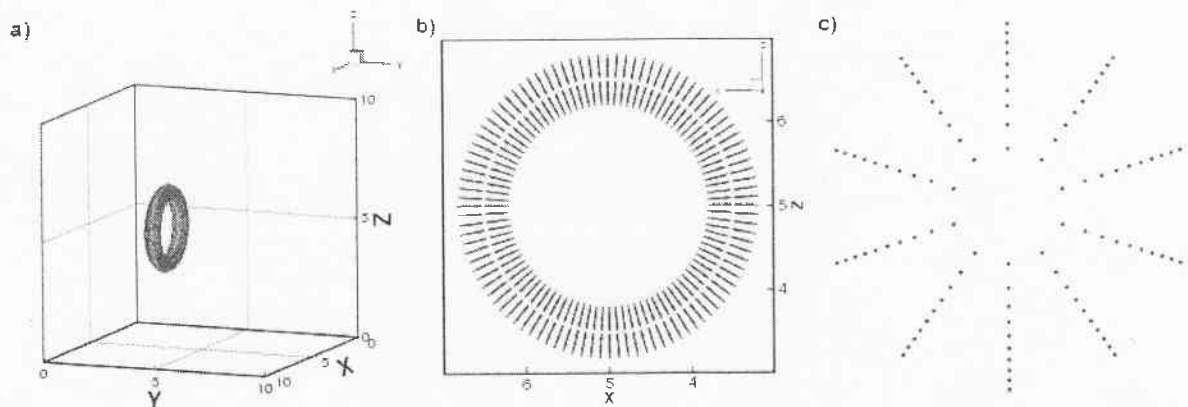


Figure 1. (a) coordinate system with the vortex ring; (b) vortex ring that was divided by one hundred slices; (c) initial position of the particles at a single slice.

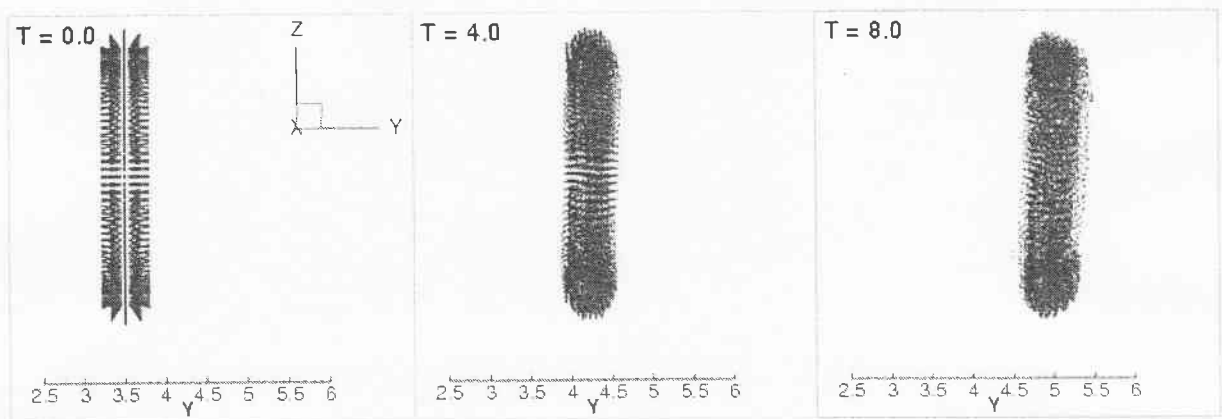


Figure 2. The sequence of the vortex ring position that are visualised by the particles. Each particle is represented by the dot.

$\vec{n}$  - means a unit vector normal to the slice). For these parameters the velocity obtained from formula (10) is  $\sim 0.187$  and it is nearly the same as the translation velocity for the ring in Figure 2. In Figure 3 we tried to simulate the motion of two rings with the same circulation that created the “vortex leap-frogging” phenomenon (Lim & Nickels 1995). We found that the evolution of two vortex rings strongly depends on their initial positions and parameters. In Figure 4 the initial position of the first ring was closer to the larger one and its diameter was smaller than in Figure 3. Qualitative changes in evolution between Figures 3 and 4 are clearly visible. The smaller ring passes through the larger one and starts to roll-up around the larger one producing the “tail”. Qualitatively the pictures resemble the experimental one published in Shariff & Leonard (1992).

#### 4. Closing remarks

It seems that the particle vortex method for 3D flow gives reasonable results and may be very useful for the simulation of the viscous 3D flow.

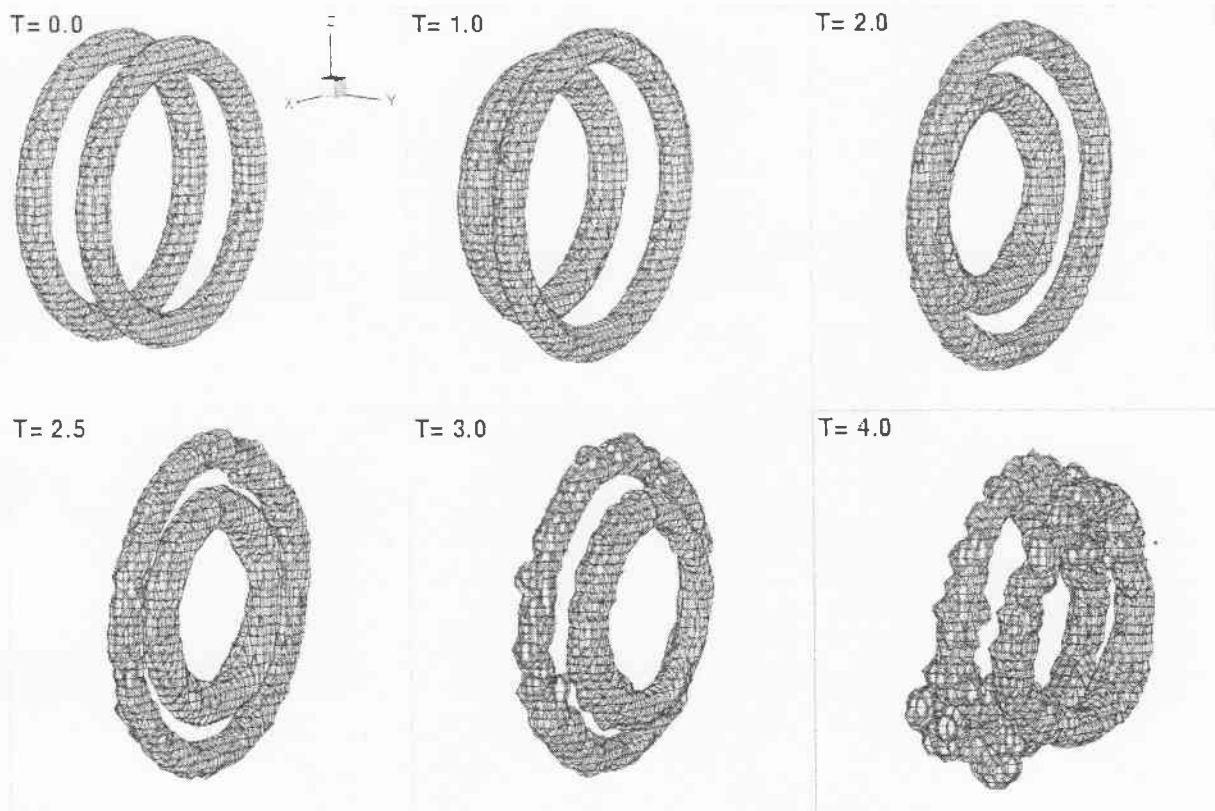


Figure 3. Time evaluation of  $|\omega|$  surface of the leap-frogging of the vortex rings, number of particles was  $2 \times 10000$  (10000 for one ring).

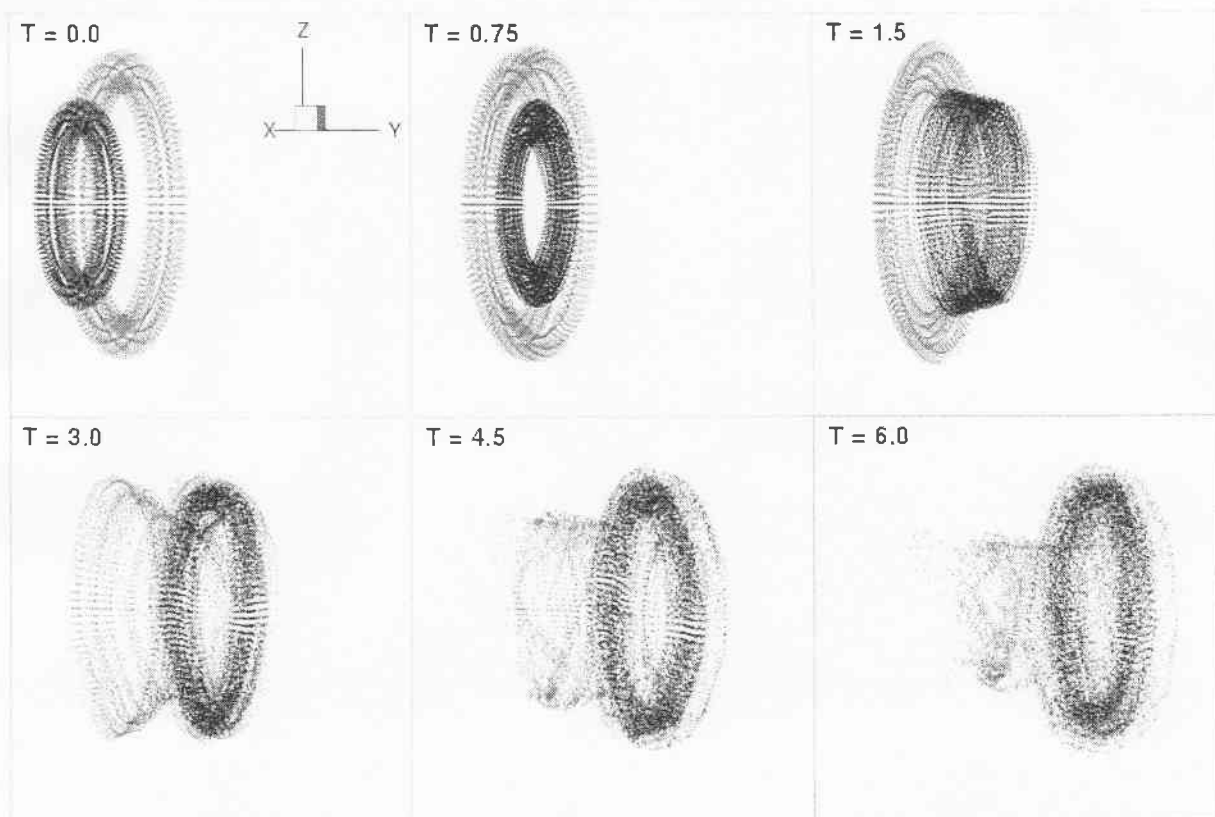


Figure 4. The sequence of the time position of the vortex particles for the motion of the two rings with the same circulation  $\Gamma = 1$ .

During our calculations we monitored the kinetic energy  $E = \int u^2 d\vec{x}$ , helicity  $H = \int \vec{\omega} \cdot \vec{u} d\vec{x}$ . For the single ring  $H \approx 10^{-7}$ . For the case of “vortex game” (Figure 3)  $H$  changed from 0 to  $H \approx 10^{-2}$ , for the case from Figure 4 final helicity was  $H \approx 10^{-4}$ . Energy for the single ring (Figure 2) changed less than 1% but for the case from Figure 4 energy changed nearly about 40%. We suspect that using a smoother redistribution of the “mass vorticity” (Cottet 2000) on the grid nodes will improve the results.

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