THE INFLUENCE OF SURFACE-TENSION EFFECTS ON USING VORTEX METHOD IN THE STUDY OF RAYLEIGH-TAYLOR INSTABILITY

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SUMMARY

The effect of surface-tension on the smoothing of irregular motion of vortices in the vortex simulation of Rayleigh-Taylor instability is shown. The irregular motion appears as an effect of short-wave disturbances the source of which is round-off error. Inclusion of surface tension allows the observation of the formation of singularities. The singularities make an infinite jump discontinuity in the curvature of vortex sheet. It is observed that for sufficiently small Atwood number and for initial amplitude perturbation large enough two singularities appear in a half period of the vortex sheet and that only one appears for greater Atwood numbers.

INTRODUCTION

Rayleigh-Taylor (R-T) instability occurs on the interface between two fluids with different densities when the less dense fluid is accelerated in the direction of the denser one [5][14]. With some simplified assumptions, the interface can be regarded as a vortex sheet, and the investigation of its evolution can be reduced to the solution of an initial-value problem. For the solution of this problem we can use a method based on boundary integral techniques. The vortex-sheet formulation described by Baker,Meiron,Orszag (B-M-O) [2] was chosen. The method was successfully used for the case when Atwood number $A=(\rho_1-\rho_2)/(\rho_1+\rho_2), \rho_1>\rho_2$ was equal one. The method failed for $0<A<1$. The linear approximation of R-T, without any stabilizing mechanism, gives an arbitrarily large growth rate of short-wave length perturbations [5]. In computations, the source of short-wave perturbations is round-off error due to finite precision arithmetic [9], and without any stabilizing mechanism it leads to the deterioration of calculations. Similar to Kelvin-Helmholtz instability one may say that the problem is ill-posed in the Hadamard sense [9],[11],[13]. To suppress the chaotic, irregular motion of vortex points in numerical study of vortex sheet evolution many tricks were introduced [12][9]. In this paper it was demonstrated that in the R-T problem the inclusion of surface- tension effects can suppress the irregular motion of vortex points. This permits one to observe the process of singularity formation. The surface tension provides a small-scale stabilizing mechanism which doesn't remove the ill -posedness of the problem. To do this a parameter δ^2 was introduced in the Cauchy-type integral to cut-off the singularity of integrand [10] [8] .In this situation, the surface tension effects caused also the lengthening of time interval in which numerical solutions could be obtained.

STATEMENT OF THE PROBLEM

It is assumed that the motion of the fluids is potential,two-dimensional and that the fluids are inviscid. A rectangular system of coordinates (x,y) is introduced. Before introducing a perturbation of the interface the fluid with density ρ_1 (ρ_2) occupies the upper (lower) half plane and $\rho_1>\rho_2$. A constant gravitational field g acts in the negative y-direction. Both ρ_1 and ρ_2 are constant and there is a density jump at the interface. The interface is described as a complex curve $z(a,t)=x(a,t)+iy(a,t)$, where "a" is regarded a Lagrangian parameter. It is assumed that the interface and the disturbance are periodic in the x-direction with wavelength λ_p. So $a \in [0,a_o]$ and an increment a to a_o gives

$$z(a+a_o,t) = z(a,t) + \lambda_p . \tag{1}$$

The normal velocity component on the interface is continuous and the tangential velocity component on the interface may have jump. So the interface can be regarded as a vortex sheet with intensity:

$$\Gamma(s,t) = (\underline{u}_2 - \underline{u}_1)\underline{s}^{\circ} = \frac{\partial\,[\Phi]}{\partial s} \tag{2}$$

where $\underline{u}_1,\underline{u}_2$ are the velocities of the upper and lower fluids at the interface, $\underline{s}^{\circ}$=unit tangential vector, s-arclength, $[\Phi]$-jump of the value of the potential across the interface, t= time.
At t=0, the fluid is assumed at rest; then the flat interface y=0 is perturbed and has the form $y(x,t)=\varepsilon(t)\cos Kx$ for t>0. In the linear approximation [5] [14] it is known that the amplitude $\varepsilon(t)$ will behave like $\varepsilon(t)=\varepsilon(0)\cosh \omega t$ where

$$\omega^2 = g\,\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\,K - \frac{\sigma}{\rho_1 + \rho_2}\,K^3 \tag{3}$$

and where σ is the coefficient of interfacial tension, $K=2\pi/\lambda$.
From (3) we can see that for $\sigma=0$, then $\omega>0$ and the interface is unstable. The growth rate for short wavelengths is unbounded. Inclusion of surface-tension effects ($\sigma\neq0$) stabilizes the perturbations shorter than a critical wavelength

$$\lambda_c = 2\pi\left[\frac{\sigma}{g\,(\rho_1-\rho_2)}\right]^{1/2}. \tag{4}$$

To put the further equations in non-dimensional form we choose λ_p as a unit of length, $\sqrt{\lambda_p/Ag}$ as a unit of time and the surface-tensions coefficient was normalized by $g(\rho_1-\rho_2)\lambda_p^2$.

THE GOVERNING EQUATIONS

By virtue of the Biot-Savart law, the velocity induced by the periodic vortex sheet at a point on the sheet can be expressed as:

$$q^* = u(a,t)-iv(a,t)= \frac{1}{2i} \fint_0^1 \gamma\,(a',t)\cot \pi(z(a,t)-z(a',t))da' \qquad (5)$$

where $\gamma(a,t)=\Gamma(a,t)(x_a^2+y_a^2)^{1/2}$ The subscript denotes a differentiation and the dash on the integral sign signifies Cauchy principle value.

The vortex sheet will be replaced by a suitable distribution of vortices (Lagrangian points) each labelled by a parameter "a". The later position of those points allow the determination of the interface shape. We define the Lagrangian point velocity on the vortex sheet as follows [4][3]

$$\frac{\partial z}{\partial t} = \frac{q_1 + q_2}{2} + \alpha \frac{q_2 - q_1}{2} \qquad (6)$$

where q_1, q_2 are the upper and lower limits of the fluid velocities on the interface, and $\alpha \in [-1,1]$ is a weighting factor. Note that for $\alpha=0$ the Lagrangian point is non-material and for $\alpha=-1(1)$ it follows the upper (lower) fluid at interface. In this way we retain some control over the positioning of the Lagrangian points. By virtue of the Sochocky-Plemelji formula [10] (see also [4])

$$q^*_{1(2)} = q^* \; (\mp) \; \frac{\gamma}{2z_a} . \qquad (7)$$

The velocity of a Lagrangian points on the interface can be expressed as:

$$\frac{\partial z^*}{\partial t} = q^* + \alpha \frac{\gamma}{2z_a} . \qquad (8)$$

To equation (8)(5) there must be added the equation for vortex-sheet strength due to baroclinic generation of vorticity. This equation is [3] :

$$\gamma_t = 2A\left(\mathrm{Re}(q_t^* z_a) - \frac{1}{2}\alpha\gamma \frac{u_a x_a + v_a y_a}{z_a z_a^*} + \frac{1}{8}\frac{\partial}{\partial a}\frac{\gamma^2}{z_a z_a^*} + \frac{1}{2}\alpha \frac{\partial}{\partial a}\frac{\gamma^2}{z_a z_a^*}\right) +$$

$$+2y_a - 2\sigma k_a + \frac{1}{2}\alpha \frac{\partial}{\partial a}\frac{\gamma^2}{z_a z_a^*} , \qquad (9)$$

where $k=(x_a y_{aa} - y_a x_{aa})/(x_a^2+y_a^2)^{1.5}$ = the curvature of vortex sheet, σ = the non-dimensional coefficient of surface tension. The time derivatives q_t^* are obtained by differentiation of (5). After substitution of these derivative in (9) we obtain:

$$\gamma_t = A \int_0^1 \gamma_t \, B \, da' + r(x,y,u,v,\gamma) \qquad (10)$$

where $B = -x_a \dfrac{\sinh 2\pi(y(a)-y(a'))}{\cosh 2\pi(y(a)-y(a'))-\cos 2\pi(x(a)-x(a'))} + y_a \dfrac{\sin 2\pi(x(a)-x(a'))}{\cosh 2\pi(y(a)-y(a'))-\cos 2\pi(x(a)-x(a'))}$

and r( ) does not depend on γ_t. Eq. (10) represents a Fredholm integral equation of the second kind and is solved iteratively.

As has already been mentioned in the introduction, the initial value problem for R-T is ill-posed. The solution within a finite time yields a singularity which is an infinite jump in the curvature. In order to regularize the problem the singularity of the integrand of the Cauchy -type integral is cut off. Components of velocity of equation (5) are now:

$$u(a,t) = -\frac{1}{2} \int_0^1 \frac{\sinh 2\pi(y(a)-y(a'))}{\cosh 2\pi(y(a)-y(a'))-\cos 2\pi(x(a)-x(a')) + \delta^2} \, da' \qquad (5a)$$

$$v(a,t) = \frac{1}{2} \int_0^1 \frac{\sin 2\pi(x(a)-x(a'))}{\cosh 2\pi(y(a)-y(a'))-\cos 2\pi(x(a)-x(a')) + \delta^2} \, da'. \qquad (5b)$$

Now the integrals in (5a,b) are not singular and if we calculate with the assistance of (5a), the acceleration u_t, v_t, eq. (10) also will not have a singular integral. In the case of weak stratification when $\rho_1/\rho_2 \longrightarrow 1$, instead of (9) there is often utilized the Boussinesq asymptotic approximation . We take $A \longrightarrow 0$ and $g \longrightarrow \infty$ in such a way that the product Ag goes to a finite limit [15]. Equation (9) takes the form (α=0):

$$\gamma_t = 2y_a - 2\sigma k_a \, . \qquad (9a)$$

Equations (8),(10) constitute the complete description of the evolution for the interface. The numerical procedure goes as follows [3]: for known $x(a), y(a), \gamma(a)$ the interface is marched forward using (8),(5) and next inhomogeneous term r( ) in (10) is calculated. Then the integral equation (10) is solved iteratively for γ_t and finally γ can also be marched in time. When we use Boussinesq approximation ((9a) we do not need to solve integral equation and γ_t is simply calculated from (9a). The time stepping is performed, as in [3] by using a fourth -order Adams-Moulton predictor-corrector scheme. All derivatives with respect to the Lagrangian parameter "a" were computed using cubic spline. All Cauchy type integrals were regularized by subtraction of integral with a known principle value [3] and integrals were calculated using trapezoidal rule on alternate sets of points [4].

NUMERICAL RESULTS

Results presented below were obtained on an IBM PC-AT in double precision (15 decimal digits) arithmetic using RMFORTRAN. The number of vortices accepted for calculations is N=120,time step $\Delta t=0.004$. Fig.1a,b shows solutions for A=1 ($\rho_1/\rho_2=\infty$,the so-called single fluid case). Fig. 1a shows a time sequence of interface profile y(x,t) and fig.1b shows a time sequence of vortex sheet strengths $\Gamma(a,t)$ vs Lagrangian parameter "a". $y(x,0)=\varepsilon\ \cos 2\pi x$,$\varepsilon=0.5/2\pi$ was the initial condition. Such an amplitude makes possible a comparision results with [2]. These single fluid case results were described precisely by B-M-O [2] and can be regarded as an "exact" solution [15]. They make a good test for the numerical program. The points on the curve indicates the positions of vortices along the interface. For α in (6) we took $\alpha=-1$. This means that vortices move like particles of upper fluid.

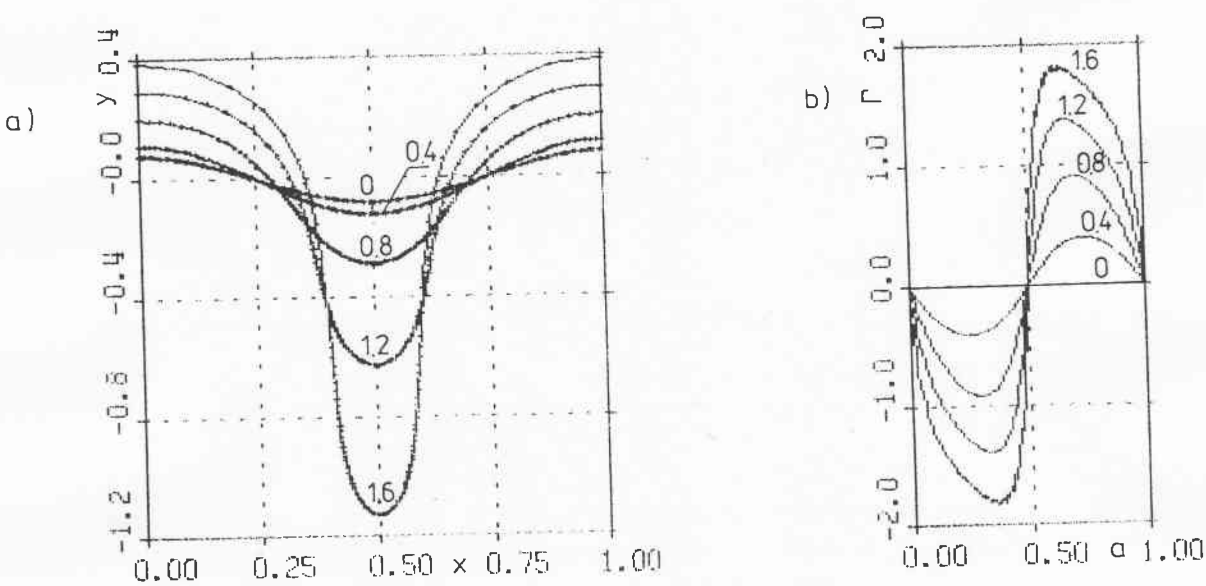

Fig.1. Numerical results for A=1,$\alpha=-1$,$\delta^2=0$,$\sigma=0$.a)interface profile y(x,t), b) vortex sheet strength $\Gamma(a,t)$

Fig.2a)b)c) show numerical results for A=0.0476 ($\rho_1/\rho_2=1.1$),$\sigma=0$ and $\varepsilon=0.1$. Fig.2a shows the time sequence of interface profiles (only half a period), fig.2b the time sequence of the curvature k(a) of the interface and fig.2c shows the time sequence of the vortex sheet strength $\Gamma(a)$. The inspection of curvature is a good means for checking the smoothness of the solutions and it shows the symptoms of the regularity loss of the solution earlier than one can see on interface profile. In fig.2b for t=0.74 one can see the irregular, noisy changes of curvature at intervals where vortex sheet strength has its maximum approximately.It is due to Helmholtz instability. Those noisy changes start little earlier and their amplitude grows in time

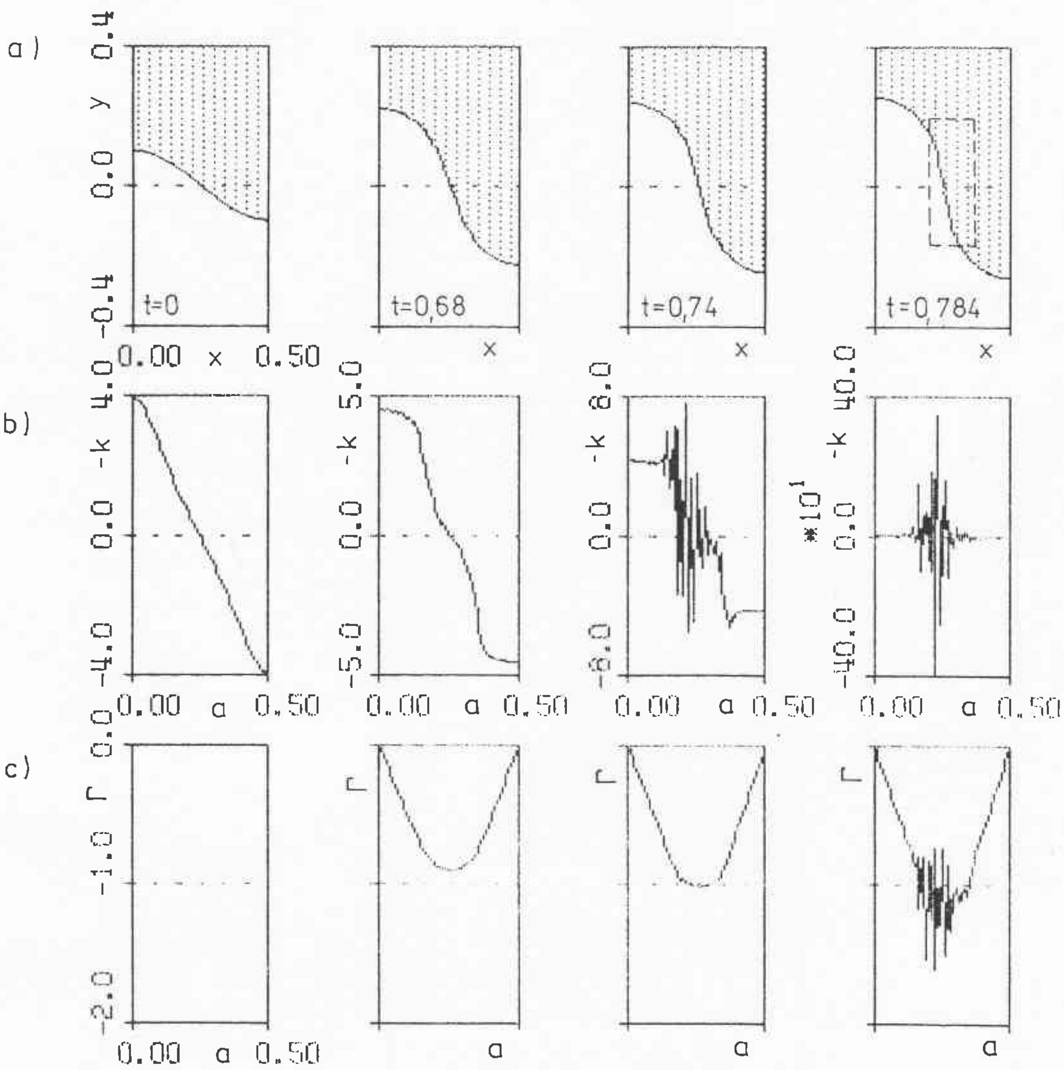

Fig.2 Numerical results for A=0.0476 (ρ_1/ρ_2=1.1), δ^2=0, σ=0, α=0, y(x,0)=0.1cos2πx, a)time sequence of interface profile y(x,t), b) curvature-k(a,t), c) vortex sheet strength Γ(a,t)

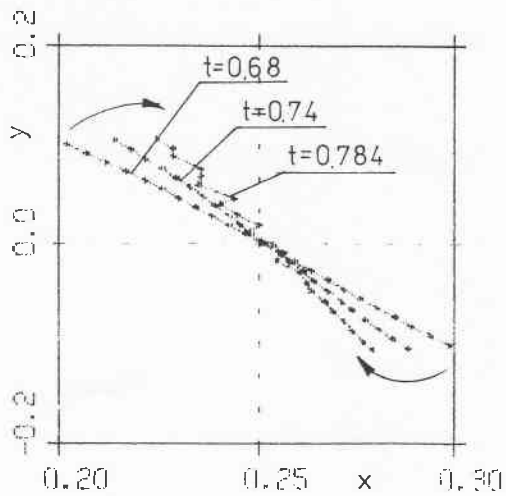

Fig.3 The time sequence of close-up fragments of interface created the same particles for A=0.0476, σ=0.

Fig.3 shows the time sequence of the close-up fragments of the interface matched by the frame in fig.2a t=0.784. These fragments are created by the same particles. For t=0.784 there can be seen distinctly the irregular positions of the vortices. Fig.4a)b) show the time sequence of curvature and strength of vortex sheet in the case when the surface-tension effects are included. For the value of coefficient of interfacial tension we took $\sigma=5*10^{-4}$.Now the noisy ,chaotic changes don't appear in the curvature.The curvature distribution for t=0.824 has two peaks and typical tips appear in distribution of vortex sheet strength.

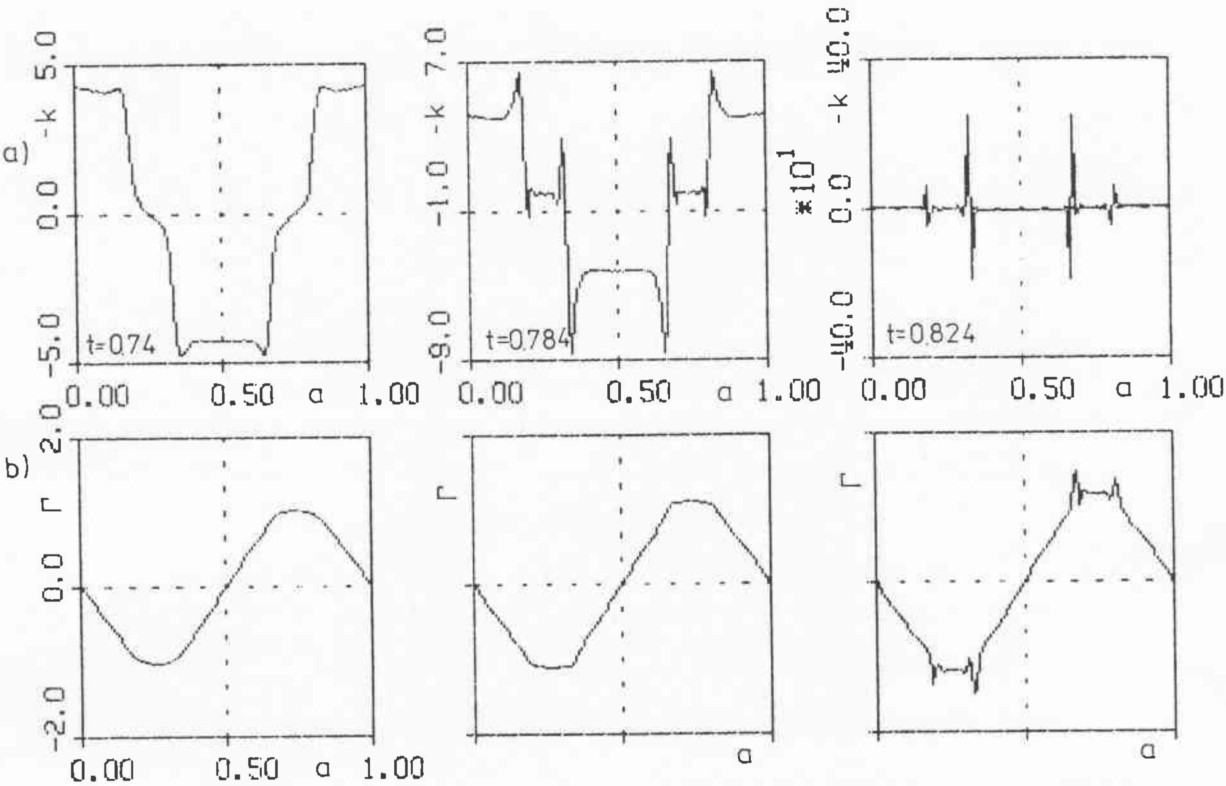

Fig.4 The time sequence a) the curvature k(a,t), b) the vortex sheet strength Γ(a,t) for for the same data as in fig.2 but with surface tension effects σ=0.0005.

Fig.5 shows the time sequence of the close-up fragments of the interface created by the same particle as in fig.3. The origination of cusps is visible and it can be related to the formation of singularities.The suppression of the irregular motion of vortices is evident. The formation of two singular points is connected with large initial amplitudes of perturbation (ε=0.1).For ε=0.01 we can observe only one singularity along half a period of the interface. Fig.6a shows the time sequence of close-up fragments of interface and figs.6b,c show examples of the curvature distribution and vortex sheet strength for time t=1.48.The results which relate to the singularity formation which I present here ,are very similar to the results presented by Krasny [9]. For larger Atwood number in spite of ε=0.1 only one singularity is originated. The singularity point moves towards the point of symmetry x=0.5. Fig.8 shows results for A=0.5. Fig.8a shows the time sequence of close-up of interface fragments and fig.8b,c the example of curvature -k(a) and vortex sheet strength Γ(a) for t=0.8.The singularity formation in solutions is the main reason the B-M-O method failed for 0<A<1. Fig.9 shows the time sequence of the profile interface for the desingularized equations for δ^2=0.1, A=0.0476, σ=0.0005, α=0. The inclusion of surface-tension effects increases the time interval for which the iteration process for γ_t converged.

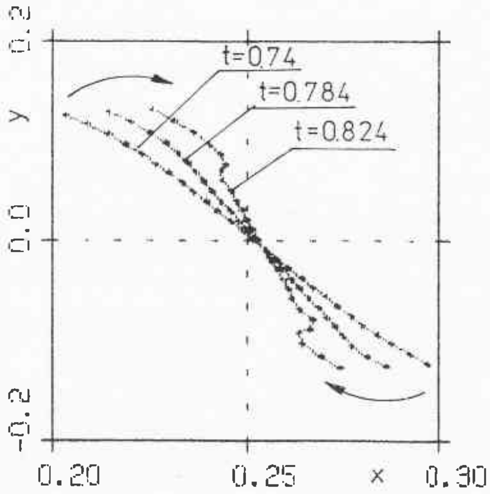

Fig.5 The time sequence of close-up interface fragments for included surface-tension effects σ=0.0005, A=0.0467, α=0, ε=0.1.

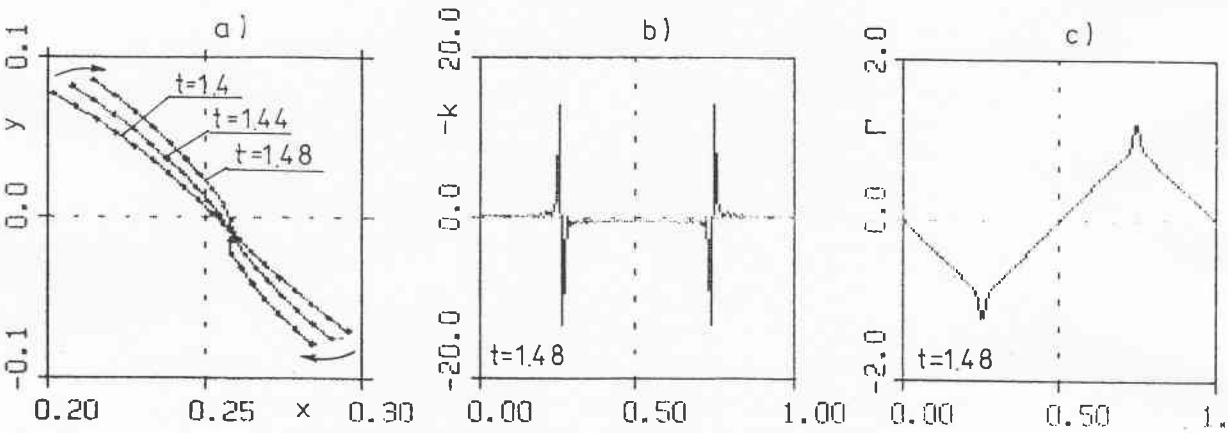

Fig.6 Numerical results for data as in fig 5 but with initial amplitude ε=0.01, a)time sequence of the close-up fragments of interface, b) curvature -k(a) for t=1.48, c)vortex sheet strength Γ(a) for t=1.48

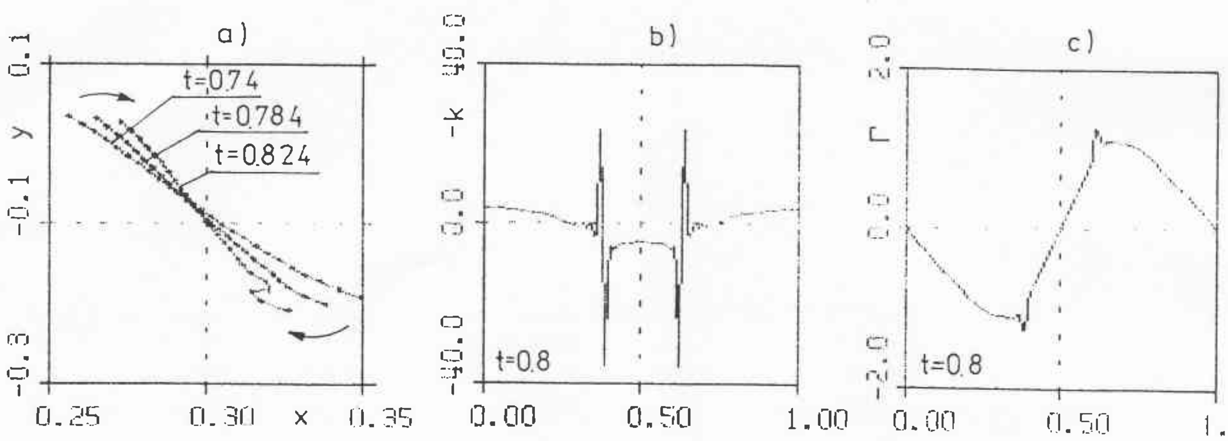

Fig.7 Numerical results for A=0.5 (ρ_1/ρ_2=3), σ=0.0005, α=-0.2, ε=0.1, a)time sequence of close-up fragments of interface b) curvature of vortex sheet -k(a) for t=0.8 c) vortex sheet strength Γ(a) for t=0.8.

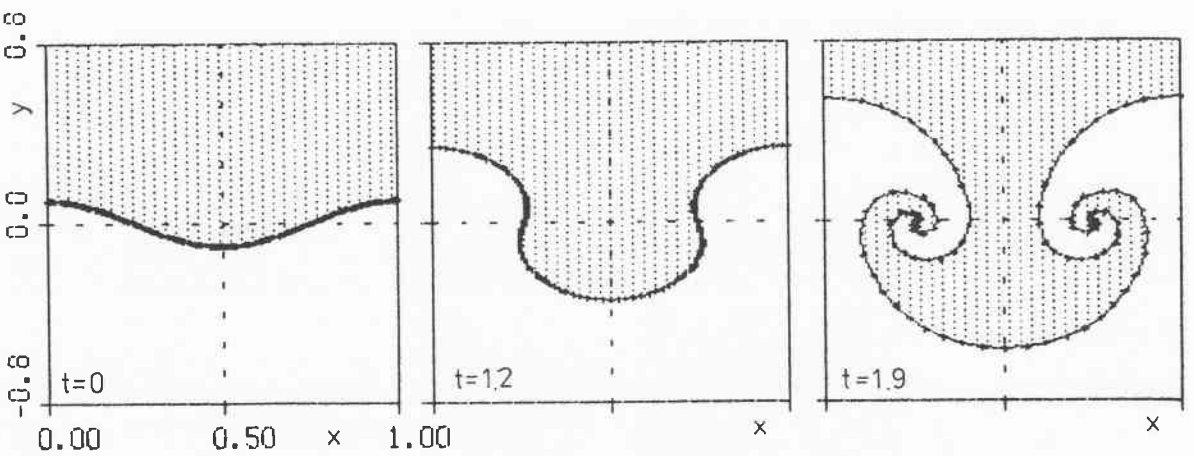

Fig.8 The time sequence of interface profile for desingularized equation δ^2=0.1,A=0.0476,σ=0.0005,α=0, ε=0.1.

Fig.9a,b show the time sequence profiles with and without surface tension for A=0 (Boussinesq approximation),δ^2=0.1,α=0,and for a more complex two-wave initial perturbation as was described by Aref [1]. Viz. $y(x,0)=\varepsilon_1\cos2\pi x + \varepsilon_2\cos6\pi x$,where ε_1,ε_2 were chosen in such a way that the initial profile had six inflection points. As can be seen in fig. 9 a (σ=0), six vortices grow from these six inflection points. But in fig. 9 b the surface-tension effect prevents the initiation of the vortex in the line y=0. We choose N=200, and the coefficient for interface surface tension σ is two times greater than in previous run viz. σ=0.001. This smoothing action of surface-tension effects is in good qualitative agreement with the results obtained by Daly in numerical study of the R-T instability by the marker-in-cell method for full Navier-Stokes equations [6].

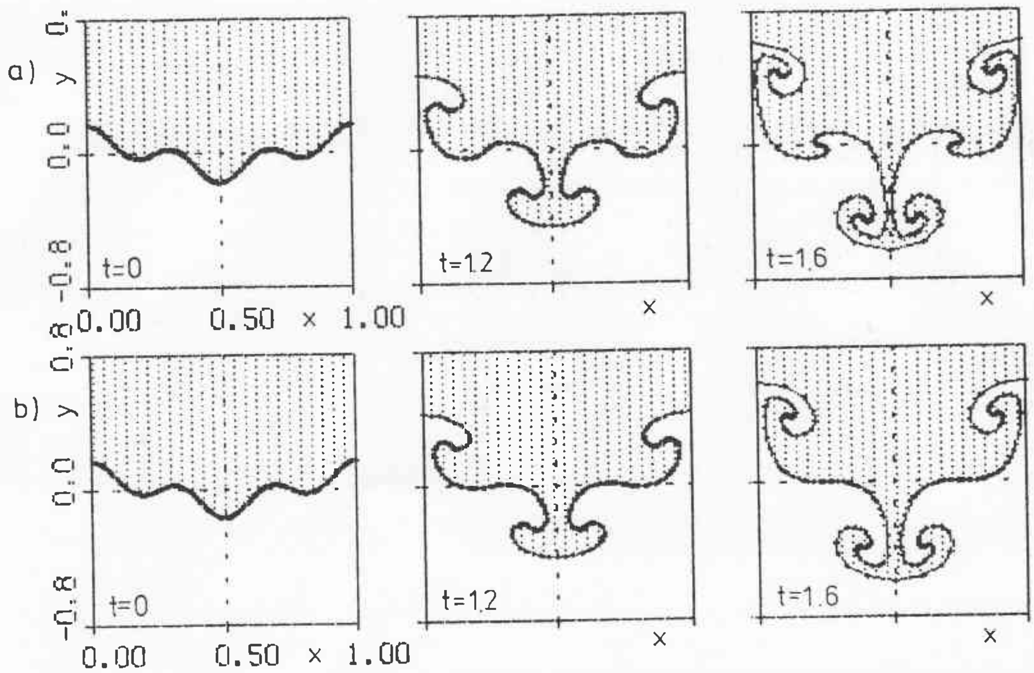

Fig. 9. Numerical results for $y(x,0)= 0.1\cos2\pi x + 0.07\cos6\pi x$, A=0, α=0, δ^2=0.1, N=200 a) σ=0, no surface-tension b) σ=0.001

CONCLUSIONS

The inclusion in the R-T problem of surface-tension effects causes the suppression of irregular motions of vortices. Singularity formations are the main reason why the B-M-O method fails for $0<A<1$. After desingularization of the equation of motion by the δ^2-parameter the surface tension also smoothes the numerical results and causes lengthening of the time interval for which the iteration process for γ_t converges.

Due to nonlinearity of the evolution equation for the vortex sheet strength after a certain time interval we obtained a nearly singular distribution of vortex sheet strength and finally the calculation broke. Alternative to direct summation method,very efficient computationally are vortex-in cell methods described by Tryggvason [15]. But as Tryggvason says "grid free method although considerably more inefficient"...than vortex in cell method.." offers a "cleaner" environment (no grid disturbances) to study such delicate questions as singularity formations and how to apply proper regularizations".

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