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Collapse of *n*-point vortices in self-similar motion

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Abstract
In this paper the initial positions of *n*-point vortices that collapse in finite time are numerically obtained. Examples of the collapsing systems of 3, 15 and 24 vortices are given. How to obtain the collapse positions of the vortices in the configuration space that are parameterized by the Hamiltonian values is described. A solution for the self-similar collapse trajectories of the vortices is derived.

(Some figures may appear in colour only in the online journal)

1. Introduction

The study of the dynamics of point vortices in a plane for an ideal incompressible fluid has been a very active research field for a long time. In 1858 Helmholtz (Helmholtz 1858) wrote the paper 'On integrals of the hydro-dynamical equations, which express vortex motion' where the well-known equations for the systems of vortices appeared (see (1) below). Kirchhoff showed in his 1877 lectures that the equation for the vortices' motion may be expressed as a Hamiltonian system (Aref *et al* 1992). In 1877 Gröbli in his dissertation studied analytically the motion of three vortices and showed that three vortices can collapse to a point in finite time (for the historical background see Aref *et al* (1992)). Unfortunately, the works of Gröbli were completely forgotten. A three-vortex system is integrable since it poses the three independent integrals that have commuting Poisson brackets (Aref 2007). Liouville's theorem of classical mechanics assures that in such a case the Hamiltonian system is integrable and can be explicitly reduced to the quadratures (Kozlov 2003). Yet in Batchelor's book (Batchelor 1967) it was written that the detail of the motion of three vortices is not evident. The analytical results of Gröbli were rediscovered by Novikov (1975) and independently by Aref (1979). The collapse of three vortices in self-similar motion was shown by Aref (1979) and Novikov and Sedov (1979). Novikov and Sedov (1979) gave special, limited examples for the collapse of four and five vortices.

The collapse of the vortices belongs to one of the most interesting problems related to the dynamics of vortices. The problem represents the great interest shown in the theoretical investigation of turbulent motion. The collapsing system of vortices changes the scales of the motion that is the fundamental feature of the turbulence. It is one of the scenarios related to the loss of the uniqueness of Euler's equations in three dimensions. We must remember that the point vortices are represented by a set of rectilinear vortex lines in three dimensions. The collapsing solution in two dimensions (2D) becomes a three-dimensional collapsing solution, despite the fact that vorticity is extended to infinity.

The point vortex model provides a useful model to many interesting physical systems (Aref 2007) and is the basis for the vortex particle method, a very efficient numerical method for the solution of the Euler as well as the Navier–Stokes equations (Cottet and Koumoutsakos 2000, Kudela and Malecha 2009). It is worth indicating that in atmospheric physics, collapsing vortices can be seen as the process that leads to the formation of a large atmospheric vortex like a cyclone.

In this paper we have demonstrated numerically the collapse of the vortices to a point in finite time for any number of vortices. To the best of the author's knowledge, it is the first time such a demonstration has taken place.

2. Equations of motion of system vortices

We start by recalling the equations of motion of the vortices system. Let us assume that there are n-point vortices on the plane with distinct positions $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, $z_k = x_k + \mathrm{i}y_k$ and circulations (intensities) $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, each $\Gamma_j \in \mathbb{R}\setminus 0$. The differential equation that describes the motion of the system of n vortices is (Newton 2001, Kochin *et al* 1965)

$$\frac{\mathrm{d}z_k(t)}{\mathrm{d}t} = v_k\big(z(t)\big) = \frac{\mathrm{i}}{2\pi}\sum_{j=1}^{n}{}' \, \Gamma_j \frac{1}{\bar{z}_k - \bar{z}_j}, \tag{1}$$

where the prime on the summation indicates omission of the term with $j = k$. For the system (1) one can prove the following identities (Kochin *et al* 1965):

$$\sum_{k=1}^{n} \Gamma_k v_k = 0, \tag{2}$$

$$\sum_{k=1}^{n} \Gamma_k \bar{z}_k v_k = \frac{\mathrm{i}}{2\pi}\sum_{k>j} \Gamma_k \Gamma_j, \tag{3}$$

$$\sum_{k=1}^{n} \Gamma_k v_k \bar{v}_k = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k>j} \frac{\Gamma_k \Gamma_j}{2\pi \mathrm{i}} \ln\left(z_k - z_j\right). \tag{4}$$

Using these identities, one can prove for system (1) the following integrals (Kochin *et al* 1965):

$$H = -\frac{1}{2\pi}\sum_{j=1}^{n}\sum_{k,\, k\neq j}^{n} \Gamma_j \Gamma_k \ln r_{jk} = \text{const.}, \qquad r_{jk} = \left|z_j - z_k\right| \tag{5}$$

$$M = M_x + \mathrm{i}M_y = \sum_{j=1}^{n} \Gamma_j x_j + \mathrm{i}\sum_{j=1}^{n} \Gamma_j y_j = \text{const.} \tag{6}$$

$$V = \sum_{k=1}^{n} \Gamma_k \left(x_k \frac{\mathrm{d}y_k}{\mathrm{d}t} - y_k \frac{\mathrm{d}x_k}{\mathrm{d}t} \right) = \frac{1}{2\pi} \sum_{k>j} \Gamma_k \Gamma_j, = \text{const.} \tag{7}$$

$$S = \sum_{j=1}^{n} \Gamma_j \left(x_j^2 + y_j^2 \right) = \text{const.} \tag{8}$$

The invariant (5) is a Hamiltonian and as shown by Kirchhoff, the system (1) can be expressed in the Hamiltonian formulation (Borisov and Mamaev 2005, Newton 2001):

$$\Gamma_k \frac{x_k}{\mathrm{d}t} = \frac{\partial H}{\partial y_k}, \qquad \Gamma_k \frac{y_k}{\mathrm{d}t} = -\frac{\partial H}{\partial x_k}. \tag{9}$$

The invariant (6) is called a linear impulse, (8) is called an angular impulse and (7) is named a virial (Aref 2007). It is possible to create one more invariant as a linear combination of (6) and (7) that plays a role in the study of the self-similar collapse of vortices (Aref 1979, Kimura 1987, Novikov and Sedov 1979)

$$L = \sum_{k>j} \Gamma_k \Gamma_j r_{kj}^2 = \sigma S - \left(M_x^2 + M_y^2 \right) \tag{10}$$

where

$$\sigma = \sum_{k} \Gamma_k = \text{const.} \tag{11}$$

Invariant L is expressed in terms of the mutual separation of the vortices r_{jk}. It is clear that when a system collapses all $r_{ij} \to 0$, so $L = 0$ is the necessary condition for the vortex collapse (Borisov and Mamaev 2005, p 66). Our interest here is to find numerically the initial positions of vortices that collapse in finite time. As was remarked in Aref (2010), due to the invariants of motion it seems that collapse contradicts the intuitive characteristic that the distance between any two vortices can never be much less than the smallest distance between any pair of vortices initially (Batchelor 1967). But as explored for $n = 3$ in Aref (1979, 2010), Novikov and Sedov (1979), and Kimura (1987), for self-similar motion and some additional conditions for intensities of vortices, the three vortices can collide in the center of vorticity in finite time. Here it will be demonstrated that such a collapse is possible for any $n \geqslant 3$.

3. Self-similar motions of *n* vortices in the plane

The dynamic of the n-vortices system is determined by the collection of n intensities Γ_k, $k = 1, \ldots, n$ and the initial position of the vortices $z_k(0)$. It is worth taking note that transformation of variable z to z' by $z' = az + b$ in (1) leads only to the division of all intensities Γ_k by $|a|$ and does not change the shape of the trajectories (Newton 2001, Synge 1949), but only the time t scales to $t/|a|^2$. The shapes of trajectories are invariant to the translations, rotations and dilatations. One can call the given configuration

$[z] = (z_1, z_2, \ldots, z_n)$ and $[z']$ as equivalent $[z] \sim [z']$ if they differ by the transformation $z_i' = az_i + b$ for some $a \in C$ and $b \neq 0$ (O'Neil 1987).

Further it will be assumed that $\sigma \neq 0$ and that the center of vorticity is shifted to the zero of the coordinate system

$$x_c = M_x/\sigma = 0, \qquad y_c = M_y/\sigma = 0. \tag{12}$$

It means that the invariant $M = M_x + \mathrm{i}M_y$ (6) is equal to zero, $M_x = 0$ and $M_y = 0$. In this paper the following definition of the self-similar collapse motion of the n vortices is assumed (see also O'Neil (1987)):

Definition 1. The system of the n vortices is in self-similar collapse if there exists complex function $\lambda(t) \in \mathbb{C}$, $\lambda(t) = \lambda_r(t) + \mathrm{i}\lambda_i(t)$ that $\mathrm{Re}(\lambda) = \lambda_r(t) < 0$ and $\lambda_i(t) \neq 0$ for each k and

$$\frac{\mathrm{d}z_k}{\mathrm{d}t} = v_k = \lambda(t) z_k, \qquad k = 1, 2, \ldots, n. \tag{13}$$

It follows from (13) that for $t = 0$ we have

$$v_k(0) = \lambda(0) z_k(0), \qquad k = 1, 2, \ldots, n \tag{14}$$

and it is easy to see we have

$$v_k(t) - v_j(t) = \lambda(t)\left(z_k(t) - z_j(t)\right), \qquad k = 1, 2, \ldots, n. \tag{15}$$

To find the solution of system (13) we introduce the new variables $(r(t), \varphi(t))$ and assume the following form of that solution (see Newton (2001), Kimura (1987))

$$z_k = z_k(0) r(t) \mathrm{e}^{\mathrm{i}\varphi(t)}, \qquad r(0) = 1, \quad \varphi(0) = 0. \tag{16}$$

After inserting (16) into (1) one obtains

$$\begin{aligned}\left(\frac{\mathrm{d}r}{\mathrm{d}t} + \mathrm{i}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)\mathrm{e}^{\mathrm{i}\varphi(t)} z_k(0) &= \frac{i}{2\pi}\sum_j^{n}{}' \frac{\Gamma_j}{r\mathrm{e}^{-\mathrm{i}\varphi(t)}\left(\bar{z}_k(0) - \bar{z}_j(0)\right)}\\ &= \frac{1}{r} v_k\left(z(0)\right) = \frac{\lambda(0)}{r} z_k(0).\end{aligned} \tag{17}$$

Separating the real and imaginary parts on both sides of equation (17) and comparing them to each other, one obtains the differential equation for $r(t)$ and $\varphi(t)$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{r}\lambda_r(0), \qquad r\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{1}{r}\lambda_i(0). \tag{18}$$

Direct integration of (18) gives:

$$r = \sqrt{2\lambda_r(0)t + 1}, \qquad \varphi = \frac{\lambda_i(0)}{2\lambda_r(0)} \ln\left(2\lambda_r(0) + 1\right). \tag{19}$$

The solution of equation (16) has the form

$$z_k(t) = \sqrt{2\lambda_r(0)t + 1}\,e^{\left(i\frac{\lambda_i(0)}{2\lambda_r(0)}\ln(2\lambda_r(0)t+1)\right)} z_k(0) = (2\lambda_r(0)t + 1)^{\frac{1}{2} + i\frac{\lambda_i(0)}{2\lambda_r(0)}}. \tag{20}$$

Solution (20) represents the logarithmic spiral. It is easy to check that when $t \to T_c$ where

$$T_c = -\frac{1}{2\lambda_r(0)}, \tag{21}$$

is called a collapse time then $z_k(t) \to 0$, $r(t) \to 0$ and $\varphi(t) \to +\infty$ when $\lambda_i(0) > 0$ or $\varphi(t) \to -\infty$ when $\lambda_i(0) < 0$. When one finds the positions $z_j(0) \neq 0$ then $\lambda(0)$ (15) and collapse time (21) can be calculated. If $\lambda_r(t) > 0$ then the vortices system expands. To change the direction of the motion we should change the sign of the circulations of all the vortices to the opposite one (Newton 2001). If the real part of $\lambda(t)$ equals zero, $\lambda_r = 0$, the vortices are in relative equilibrium and the systems rotate as a solid body. In this case the collapse time is infinite, $T_c = \infty$.

Putting the solution $z_k = z_k(0)\,r(t)\,e^{i\varphi(t)}$ to the Hamiltonian (5) one obtains

$$H(t) = H(0) - \frac{1}{2\pi}\ln|r(t)|\sum_{k,\,k\neq j}^{n}\Gamma_j\Gamma_k. \tag{22}$$

The Hamiltonian (5) during the self-similar motion will conserve, when the virial $V = 0$.

4. Algebraic equation for the initial position of collapse vortices

From (13) follows that for the collapse of vortices in self-similar motion we have

$$v_j z_k = v_k z_j, \qquad k, i = 1, \ldots, n \quad k \neq j. \tag{23}$$

O'Neil (1987) proved that there are only $n - 3$ independent equations of the form (23). Specifically, he proved that if (23) is true for example for $j = 4, 5, \ldots, n$ then it is true also for $j = 2, 3$.

The configuration space of n vortices needs $2n$ real numbers. To find the collapse position we can use only $n - 3$ equations (23). But due to the fact that we assume that the centre of vorticity is in the origin of the coordinate system we have $M_x = 0$ and $M_y = 0$. O'Neil also proved, using the identities (2) and (3) that if the virial is equal $V = 0$ then so is the angular impulse $S = 0$. From (10) it is evident that invariant L also equals zero.

From what was said above follows that the collapse configuration of vortices can be determined by the common zeros of the functions: $M_x = 0$, $M_y = 0$, $S = 0$ and $2(n - 3)$ equations for real and imaginary parts of the complex functions $v_1 z_k - v_k z_1 = 0$. As suggested by O'Neil (1987), one can expect that the collapsing vortices lie on some curves. To get the particular positions of the collapse vortices on those curves we must use a trick. To close the algebraic system we need three additional equations. To reduce the number of unknowns we fix one of the vortex coordinates, $z_n = (x_n, y_n)$ hoping that the rest of the vortices will arrange themselves to fulfil conditions for a collapse. Additionally we include to the system's real part of the identity (2) Re $\left(\sum_{i=1}^{n}\Gamma_i v_i\right) = 0$. This identity does not provide any information about the structure of the collapse position of the vortices but still should be true for any vortices system. The following non-linear system of algebraic equations was assumed:

$$f_1 = \operatorname{Re}[v_1 z_2] - \operatorname{Re}[v_2 z_1] = 0 \tag{24}$$

$$f_2 = \operatorname{Im}[v_1 z_2] - \operatorname{Im}[v_2 z_1] = 0 \tag{25}$$

$$\ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f_{n-6} = \operatorname{Im}[v_1 z_{n-3}] - \operatorname{Im}[v_{n-3} z_1] = 0 \tag{26}$$

$$f_{2n-5} = M_x = 0 \tag{27}$$

$$f_{2n-4} = M_y = 0 \tag{28}$$

$$f_{2n-3} = S = 0 \tag{29}$$

$$f_{2n-2} = \operatorname{Re}\left(\sum_{i=1}^{n} \Gamma_i v_i\right) = 0. \tag{30}$$

Thus a $2n - 2$ non-linear system of equations with fixed $z_n = (x_n, y_n)$ is solved using the Newton method. To succeed in a numerical solution of an algebraic non-linear system of equations one should provide a good guess for the initial solution (starting point). An unconstrained optimization procedure—the Levenberg–Marquart algorithm—was used to find that first approximation by minimizing the function

$$F(x_1, y_1, \ldots, x_{n-1}, y_{n-1}) = \sum_{i=1}^{2n-2} f_i^2. \tag{31}$$

Through trials in arranging the initial position of the vortices for the optimization procedure, one finds the minimum of function $F(x_1, y_1, \ldots, x_{n-1}, y_{n-1})$ that should be equal approximately to $F \approx 10^{-20}$ or smaller and that points $x_1, y_1, \ldots, x_{n-1}, y_{n-1}$ are used as a first guess for the Newton procedure. To check the quality of the solution one should verify if every z_i is different $(z_i \neq z_j,\ \ i \neq j)$ and not equal to zero. The value of $\lambda(0) = v_i(0)/z_i(0)$ has to be the same for each $i = 1, \ldots, n$. The invariants M_x, M_y and S should be equal to zero with working numerical precision. It is interesting to ask what the curves where the collapse positions of the vortices lie look like. We can obtain some tentative picture of these curves in the following manner. For the particular solution $x_1, y_1, \ldots, x_n, y_n$ the corresponding Hamiltonian $H(z(0)) = H_0$ can be calculated. By replacing equation (30) by the Hamiltonian $H_\Delta = H_0 + \Delta H = \text{const.}$ where ΔH is an arbitrarily small number, one can find the new position of collapsing vortices for this value of Hamiltonian H_Δ. By enlarging and decreasing the Hamiltonian one can obtain the set of the points which are joined by the lines given by the curves where the collapse vortices lay. All calculations were done by *Mathematica*®, v9.01. *Mathematica* has the following required procedures: *FindMinimum*—to minimise the function (31), *FindRoot* to solve the system of equation (24) by the Newton method and *NDSolve* to solve the system of differential equation (1). It is very important to carry the calculations with sufficient digital precision. Mainly high numerical precision is needed to find the minimum of function (31). In *Mathematica* the digital precision is fixed by passing to the procedure parameter that is called 'WorkingPrecision'. Using *WorkingPrecision* → *w* causes calculations to be done using numbers with *w*-digit precision (Ruskeepää 2009, p 409). Further, the values for H and T_c will be represented only by 6 digits after the decimal point.

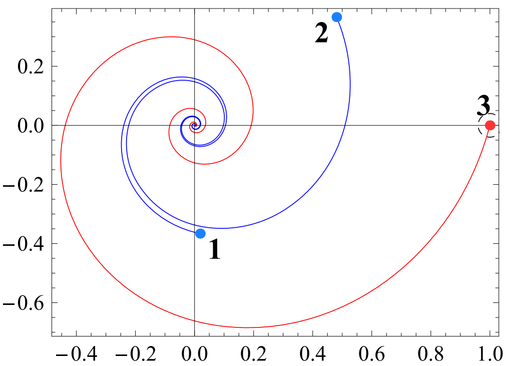

Figure 1. Collapse of three vortices: $(\Gamma_1, \Gamma_2, \Gamma_3) = (-2, -2, 1)$, $H(0) = -0.038\,1743$, $T_c = 2.739\,534$. The fixed third vortex had coordinates $(x_3, y_3) = (1, 0)$.

5. Numerical examples

5.1. Collapse of three vortices

To clarify our numerical procedure and check the numerical algorithm, we begin from the well-known problem of a collapse of three vortices (Aref 2010). According to our algebraic system (24) the collapse system of three vortices is determined only by $S = 0$, $M_x = 0$ and $M_y = 0$ and $\mathrm{Re}\left(\sum_{k=1}^{n} \Gamma_k v_k\right) = 0$. As a fixed position for the third vortex $(x_3, y_3) = (1, 0)$ was chosen. The circulations of the vortices were $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3) = (-2, -2, 1)$. The collapse position together with the collapse trajectories are presented in figure 1. The fixed vortex position $(x_3, y_3) = (1, 0)$ is marked in figure 1 by the dashed-line circle. The presented trajectories were obtained using the formula (20) as well as by solving the equations (1) using *NDSolve* with *PrecisonGoal* and *AccuracyGoal* equal to *MachinePrecision* (see Ruskeepää (2009)). The difference between these two solutions is smaller than 10^{-13}.

WorkingPrecision = 32 was chosen. The critical time was $T_c = 2.739\,534$, $H(0) = -0.038\,1743$. The minimum of the function $F(x_1, y_1, x_2, y_2)$ (31) was $\sim 10^{-43}$. After using the *FindRoot* the center of vorticity x_c and y_c and the angular impulse S had the value $\sim 10^{-33}$.

Next the Hamiltonian $H_\Delta = H(0) \pm \Delta H$ where $\Delta H = 0.0001$ was used instead of equation (30). That gave the new collapse positions for the vortices. The successive change of Hamiltonian with Δ permitted drawing on the plane the curves for the collapse position of the vortices. The numerical results were presented in figure 2 (on the left). When the Hamiltonian H was decreased to $H_{kr2} \approx -0.220\,6356$, the vortices numbered 1 and 2 were moved to location $1'$ and $2'$. Vortices $1'$, $2'$ and 3 lay on a straight line. The enlargement of Hamiltonian $H(0) = -0.038\,174$ to $H \approx 10^{-8}$ caused the displacement of vortices 1 and 2 to the position $1''$ and $2''$. The vortices $1''$, $2''$ and 3 create an equilateral triangle. For an equilateral triangle and $V = 0$ the Hamiltonian is exactly equal to zero. The positions $1'$, $2'$, 3 and $1''$, $2''$, 3 formed the only relative equilibria which is a well-known fact for the three vortices problem (Aref 2010, Newton 2001, p 73). In figure 3 the dependence of collapse time on Hamiltonian values is presented. One can notice that the curve $T_c(H)$ is very steep when H is approaching

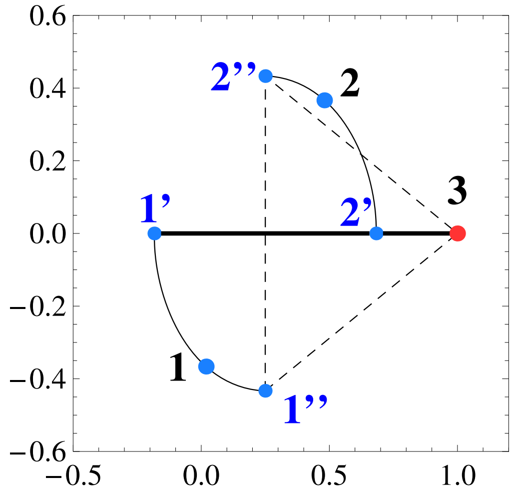

Figure 2. The curves of the collapse position of vortices while the Hamiltonian is changed from $H_1 = -0.220\,6356$, the points $(1', 2')$ to $H_2 \approx 10^{-8}$ for the point at points $(1'', 2'')$. Vortex number 3, $(x_3, y_3) = (1, 0)$ was fixed.

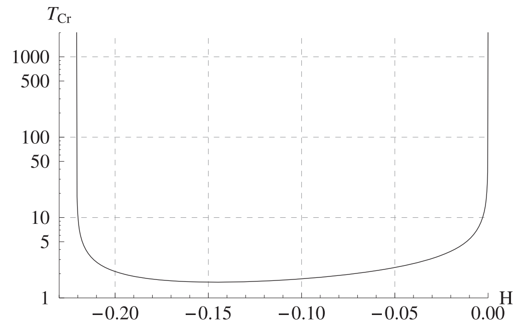

Figure 3. The dependence of the collapse time T_c on the value of Hamiltonian H.

the relative equilibria position. For relative equilibrium the collapse time is $T_c = \infty$. The minimal collapse time was $T_{c_{\min}} = 1.570\,796$ with $H\left(T_{c_{\min}}\right) = -0.145\,8401$. Trajectories for the minimal collapse time $T_{c_{\min}}$ are presented in figure 4.

5.2. Collapse of 15 vortices

Figure 5 presents the collapse trajectories for 15 vortices in the time interval $\left[0, T_c - 0.0001\right]$, $T_c = 33.901\,983$. Taking some trials from the *FindMinimum* procedure the *WorkingPrecision* = 200 was chosen. Trajectories were obtained using the self-similar solution (16). This solution was checked by solving the differential equation (1) using *NDSolve*. *PrecisionGoal* and *AccuracyGoal* were chosen equal to 16. The difference between these two solutions was smaller than 10^{-13}. The collapse time was $T_c = 33.901\,983$ and the Hamiltonian had the value $H(0) = -0.464\,567\,78$. We repeated the procedure that was described in subsection 5.1 to obtain the curves of the collapse positions of the vortices. The Hamiltonian

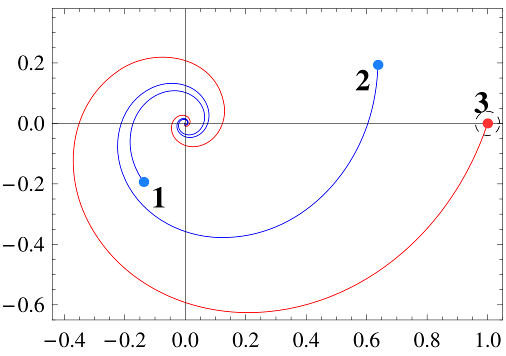

Figure 4. Trajectories for the minimal collapse time and with the same intensities as in figure 1; $T_c = 1.570\,796\,33$ and $H\,(0) = -0.145\,8401$.

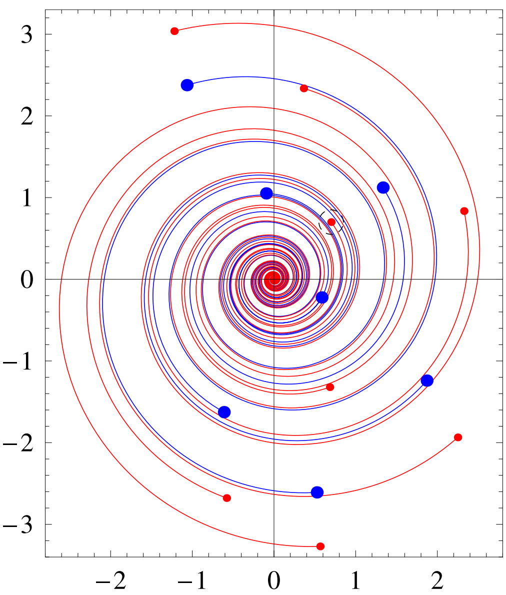

Figure 5. Collapse of the 15 vortices; $\Gamma_{(1\ to\ 7)} = -1$, $\Gamma_{(8\ to\ 15)} = 1/2$, $H\,(0) = -0.464\,5678$, $T_c = 33.901\,983$. The larger diameter points indicate the negative vortices $\Gamma = -1$, smaller diameter points mark the positive vortices with $\Gamma = 1/2$.

was changed from $H_1 = -0.480\,527\,78$ to $H_2 = -0.388\,927\,78$. An increment for the Hamiltonian $\Delta H = 0.0001$ was used. The curves of the collapse positions are presented in figure 6. The minimum of the function $F(x_1, y_2, \ldots, x_{14}, y_{14})$ along the curves was equal to 10^{-400} and the invariants S, M_x, M_y were equal to 10^{-100}. The curves started from the thick points with H_1 that were approximately in the relative equilibrium. The numerically obtained collapse time there was $T_{c_1} = 1141.8582$. The crosses on the curves mark the initial locations of the vortices that were used in figure 5. The curves ended for $H_2 = -0.388\,827$ and the

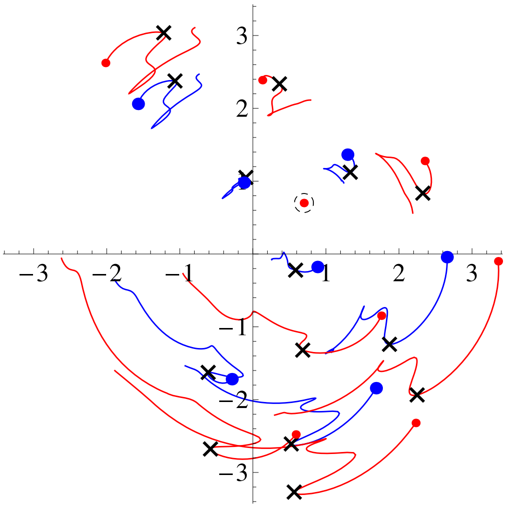

Figure 6. The curves for the locations of the collapse position of the vortices when the Hamiltonian is changed from $H_1 = -0.480\,5278$ to $H_2 = -0.388\,827$. The crosses on the curves mark initial collapse positions for the trajectories present in the figure 5.

collapse time $T_{c_2} = 30.102\,991$. Continuations of the curves were impossible due to a sudden drop in the solution's accuracy. The angular impulse S changes from $\sim 10^{-100}$ to ~ 0.0001. We see that the curves that represent the zeroes of the algebraic equations (24) for different Hamiltonian values have very irregular shapes. The curves change directions sharply. The irregular shapes of the curves indicate the very complex form of the function F (31). That is the reason for the high value of the *WorkingPrecision* parameter in the *FindMinimum* procedure.

In figure 7 the graph of collapse time T_c versus the Hamiltonian values H is presented. Once again we see that the curve is very steep when the value of H approaches the relative equilibrium. Having the dependence of collapse time on Hamiltonian values one can choose the trajectories for any collapse time, for example for the minimal collapse time. The minimal collapse time was $T_{c_{\min}} = 20.005\,039$ and is marked in figure 7 by the thick point.

5.3. Collapse of 24 vortices

The last example relates to the system of 24 vortices. Three different values were used for the intensities Γ_i: Γ_1 to Γ_{11} were equal to -1 , Γ_{12} to Γ_{18} were equal to $1/42(-77 - \sqrt{1309})$ and Γ_{19} to Γ_{24} were equal to $1/15(11 - \sqrt{1309})$. The $\sigma = -(7/30)(11 + \sqrt{1309}) \approx -11.008\,691$. To go through some trials *WorkingPrecision* = 240 was chosen. The obtained minimum of function $F(x_1, y_1, \ldots, x_{n-1}, y_{n-1})$ (see (31)) was $\sim 10^{-450}$. The values of M_x, M_y and S were numerically equal to zero. The collapsing trajectories, based on (16) are presented in figure 8. Due to the strong interactions of the vortices near the singular point the *NDSolve* procedure gave approximately the same results as (16) only for the time $0 \leqslant t \leqslant T_c - 0.15$ (*PrecisionGoal* and *AccuracyGoal* were chosen equal to 16). The collapse time was $T_c = 3.036\,775$, the Hamiltonian had the value $H(0) = -5.904\,073$. The fixed position of the vortex assumed during the solution of equations (24) is marked by the dashed-line circle, $x_{24} = -0.156\,7927$ and $y_{24} = -1.491\,7828$.

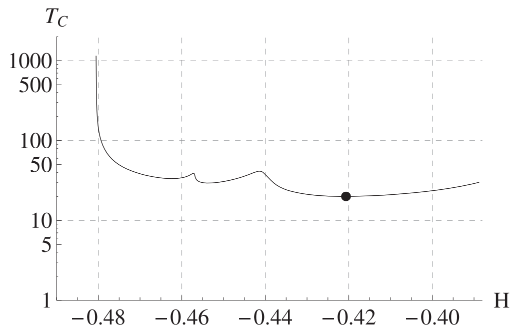

Figure 7. Dependence of the collapse time on the Hamiltonian. The thick point marks the minimal collapse time $T_{c_{\min}} = 20.005\,039$, $H\left(T_{c_{\min}}\right) = -0.420\,7278$.

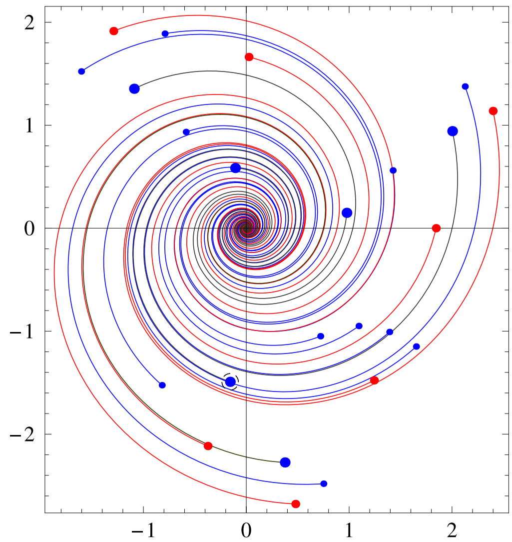

Figure 8. Collapse of the 24 vortices; $\Gamma_{(1\ to\ 11)} = -1$, $\Gamma_{(12\ to\ 18)} = 1/42(-77 - \sqrt{1309})$, $\Gamma_{(19\ to\ 24)} = 1/15(11 - \sqrt{1309})$, $H(0) = -5.904\,073$, $T_c = 3.036\,775$.

In figure 9 the curves for the collapse positions of the vortices is presented. The Hamiltonian was changed from $H_1 = -6.953\,037\,86$ to $H_2 = -5.900\,6628$ with steps of $\Delta H = \left(|H_1| - |H_2|\right)/1000$. The thick point marks the relative equilibrium. In those points the collapse time was $T_c \simeq 2 \cdot 10^4$, $H_1 = -6.953\,0379$. Invariants M_x, M_y and S along the curves were zero. The curves end with $H_2 = -5.900\,663$ and $T_c = 3.276\,088$. It was impossible to continue the curves by increasing the Hamiltonian H_2 due to the sudden loss of solution accuracy. The value of S changed from $\sim 10^{-400}$ to ~ 0.0001.

In figure 10 the dependence of the collapse time on the Hamiltonian values is presented. The minimal collapse time was $T_{c_{\min}} = 2.256\,24$ with $H = -6.258\,717$. In figure 11 the

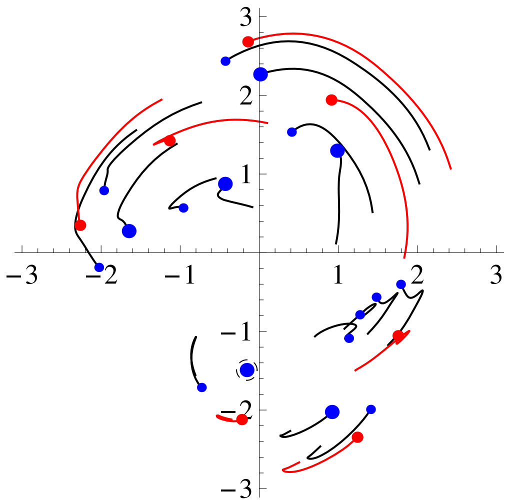

Figure 9. The curves for the locations of the collapse position of the vortices when the Hamiltonian is changed from $H_1 = -6.953\,0379$ to $H_2 = -5.900\,663$. The thick points mark relative equilibria $H_1 = -6.953\,0379$. The fixed vortex is marked by the circle, $(x_{24}, y_{24}) = (-0.156\,793, -1.491\,783)$; $T_c = 3.036\,775$. The points largest in diameter mean the vortices with negative circulation $\Gamma_{(19\ to\ 24)} = -1/15\,(11 + \sqrt{1309})$, the medium-sized points mean vortices with positive circulation $\Gamma_{(12\ to\ 18)} = 1/42\,(77 + \sqrt{1309})$ and the smallest points mean the vortices with negative values $\Gamma_{(1\ to\ 11)} = -1$.

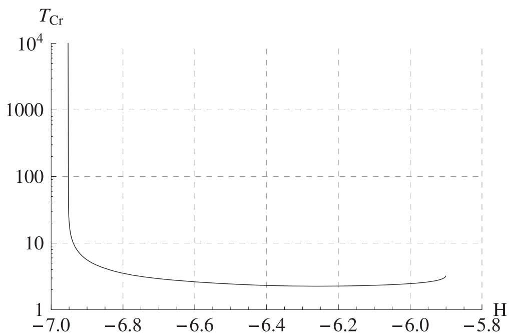

Figure 10. Dependence of the collapse time on Hamiltonian values for data as in figure 8.

solutions are presented for that minimal collapse time. One can observe that the smaller number of the spiral turns around the origin of the coordinate system.

It is also interesting to see the instantaneous distribution of the streamlines induced by the collapsing vortices. The relations between the stream function $\psi(x, y)$, vorticity ω and velocities (u, v) are

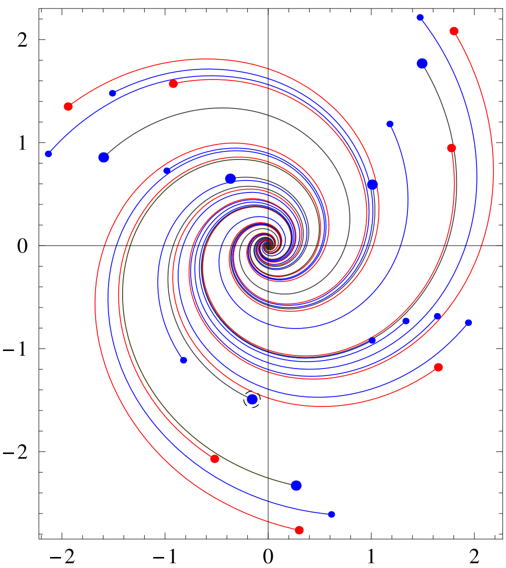

Figure 11. Trajectories with minimal collapse time for the intensities as in figure 8; $T_{c_{\min}} = 2.25624$; $H = -6.258\,717$.

$$\Delta\psi\,(x,\,y) = -\omega\,(x,\,y) \tag{32}$$

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}. \tag{33}$$

The vorticity can be approximated by the sum of the delta Dirac measures

$$\omega\,(x,\,y) = \sum_{j=1}^{n}\Gamma_j\delta\,(x - x_j)\,\delta\left(y - y_j\right). \tag{34}$$

With the stream function ψ one can express as a sum of stream functions for each individual vortex

$$\psi\,(x,\,y) = \sum_{j=1}^{n}\psi_j\,(x,\,y), \qquad \psi_j\,(x,\,y) = -\frac{\Gamma_j}{2\pi}\ln\sqrt{(x - x_j)^2 + \left(y - y_j\right)^2}. \tag{35}$$

The instantaneous pictures of streamlines for $t = 0$, $t = 0.3T_c$, $t = 0.9T_c$ and $t = 0.99T_c$ are presented in figure 12. The thick red points indicate the vortices with the positive intensities, the blue points mean the negative vortices. The black square in the middle of the frames marks the center of vortices (0, 0). To visualize the dynamics of the changes in the distribution of streamlines, the streamline $\psi = 0$ was drawn using a thick line. Dark gray smudges around the vortices indicates a negative gage pressure smaller than −3. The gage pressure was calculated using the Bernoulli law

$$p = -\frac{|v|^2}{2}. \tag{36}$$

The smudges around the vortices are irregular and the shape of the isoline $\psi = 0$ is deformed strongly. This indicates interactions between the vortices. As vortices go to the collapse point the smudges merge but still in the frame $T = 2.030\,52$ we see the white places inside of the

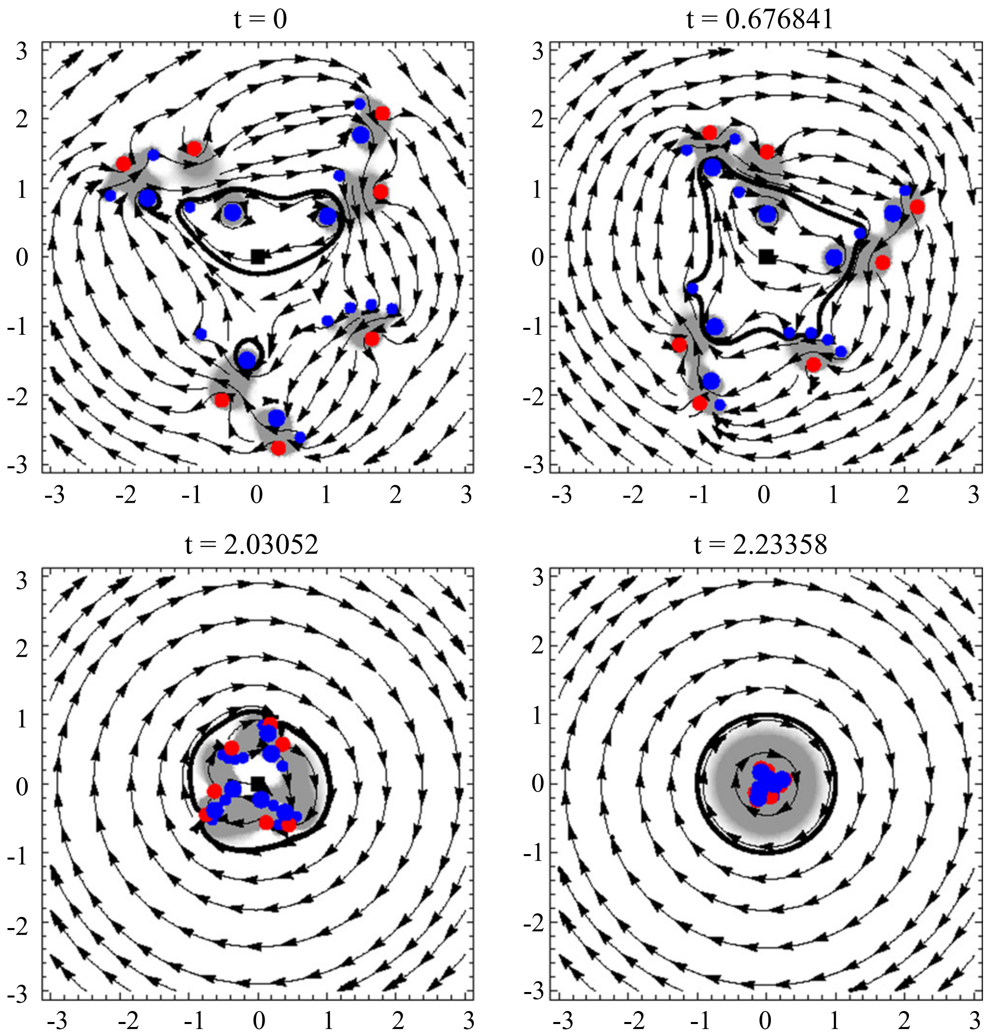

Figure 12. The instantaneous pictures of streamlines of collapse vortices for $t = 0$, $t = 0.3T_c$, $t = 0.8T_c$ and $t = 0.99T_c$. The thick line marks the streamline of zero value, $\psi = 0$, $T_{c_{\min}} = 2.256\,24$. The dark gray color means negative pressure of less than -3.

area round by the vortices ('the silence zone'). In these places the velocity is nearly zero. At the frame $T = 0.99T_c$ the streamlines are circles as if they are created by the single vortex.

Due to fact that invariant $V = 0$ and $\sigma \neq 0$ one can write

$$\sum_{k>j} \Gamma_k \Gamma_j = \left(\sum_{k=1}^{n} \Gamma_k \right)^2 - \sum_{k=1}^{n} \left(\Gamma_k \right)^2 = 0, \tag{37}$$

which means that collapse vortices do not annihilate their intensities in the collapse center and the resulting vortex has an absolute value larger than of any of the vortices that took part in the collapse process $\Gamma_R = \sqrt{\sum_{k=1}^{n} \Gamma_k^2}$ (Novikov and Sedov 1979).

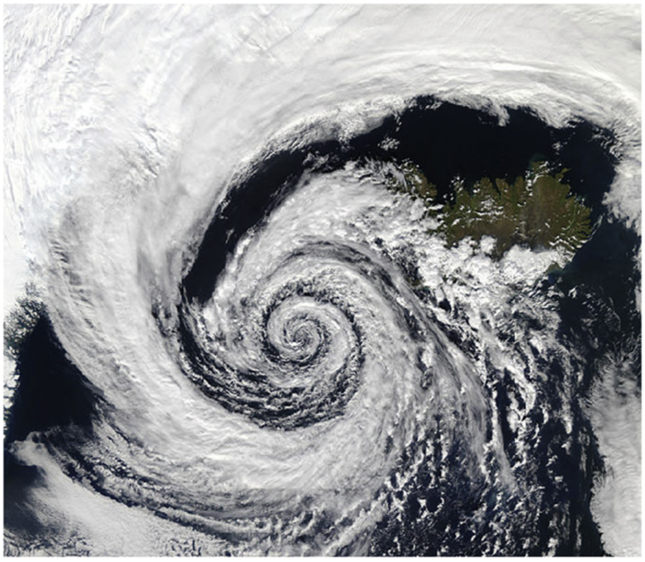

Figure 13. Polar cyclone on the southeastern coast of Greenland on 4 September 2003. Image courtesy of NASA (Descloitres 2003).

It is worth indicating the relevance of the collapse vortices problem with the meteorological phenomenon of cyclone evolution. The similarity is striking when we compare the spirals of collapse trajectories with vortices of the low-pressure system swirls of the cyclone in figure 13. The cyclone is created by the low regions of low atmospheric pressure that can be identified as vortices.

6. Concluding remarks

It was numerically demonstrated that the system of finite numbers of linear vortices under suitable initial conditions can collapse to a point with finite time. The first condition for the collapse of vortices is $V = 0$. This guarantees the constant value of the Hamiltonian H during the self-similar motion. The shift of the vorticity center to the origin of the coordinate system ($M = 0$) causes $S = 0$ and $L = 0$. Using the features of the self-similar motion of collapse vortices one can build the system of algebraic equations for finding the initial collapse positions of the vortices. The collapse vortices relate to a widely discussed problem of the loss of uniqueness of Euler's equation for inviscid flow (Wayne 2011), the transition to turbulence and stochastization (Novikov 1980). It is well-known that under very little regularity of the initial vorticity it can prove the existence of the global, regular solutions of the Euler equation in 2D (Gallagher and Gally 2005). The singular initial conditions given by the distribution of point vortices may lead to the collapse phenomenon and the loss of the uniqueness. But it is known that the point vortex approximations of the initial vorticity distribution converge to the 2D Euler equations (Goodman *et al* 1990). The regular solution and the singular one that lead to the non-uniqueness seem to be not far from each other. The 2D point-vortex model is a very useful, though simple, approximation to many important physical systems (Wayne 2011, Novikov 1980) and provides interesting physical interpretations to a number of mathematical concepts and theorems (Aref 2007). The model seems to be relevant for modeling the Earth's atmospheric motion, particularly examples such as cyclones (Descloitres 2003).

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