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STUDY OF THE "NEGATIVE TEMPERATURE" EFFECT IN THE VORTEX POINT SYSTEM BY THE VORTEX-IN-CELL METHOD

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Abstract: *The " negative temperature" effect for the point vortex system in the square box with a solid boundary was investigated. Onsager's hypothesis that at a high energy state the vortices with the same sign with cluster was verified. During the merger process of the coherent vortex structures, filamentation of the vorticity field was observed. For the calculation, the "vortex-in-cell" method was used. For the visualisation of the flow and dynamic of the vorticity field, vortex particles were used.*

1. INTRODUCTION

The appearance of coherent structures is one of the most striking features of two-dimensional turbulence. In 1949 Onsager in his famous paper [14] was the first who tried to explain of the formation of that large-scale vortex structure by statistical theory. One can assume that a large number of suitably chosen point vortices can approximate the motion of the Euler equation of fluid motion [3]. But the motion of a such point vortex system can also be described by the Hamiltonian system. The complete motion is specified by the individual location (x_n,y_n) of the particles. Therefore, the phase space of this system is simply the configuration space itself. When the motion moves into the finite space, then the phase space is also finite. Since the phase space is bounded, the number of accessible steps per unit energy cannot increase indefinitely. That is, at some point while energy is being increased, the rate of change of the entropy becomes negative. With the definition of temperature being the inverse of the rate of change entropy versus energy, we have a "negative temperature state" beyond a threshold value of energy. As a result the vortices of the same sign will tend to cluster [14].

The main goal of the present paper was to study the behaviour of the point vortex systems for different "energies". Besides the limit set of vortices, we are also interested in transition processes. It is known that the interactions between the vortices lead to vortex mergers and pairings. Those processes produce intense filaments of vorticity [5]. For numerical simulation we used the vortex-in-cell method. This method, opposite to the direct integration method, permits us to introduce into the calculation a relatively large number of vortex particles (40000). The patches, which are created by a large number of vortex particles can be regarded as continuous and incompressible. To some extent we have insight into the behaviour of the continuous field of two–dimensional vorticity.

In spite of the fact that all real flows are three-dimensional ones, it is thought that research capabilities of vortex evolution in two-dimensional space could help to understand the nature of vorticity in three dimensions.

2. THE EQUATIONS OF MOTION

The Euler equations of motion in Cartesian coordinates (x,y) for inviscid flow in domain D limited by boundary ∂D, rewritten in terms of vorticity and stream function, have the structure [1]:

$$\begin{aligned} &\frac{\partial \omega}{\partial t} + \nabla \cdot (\mathrm{u}\,\omega) = 0 \\ &\Delta \psi = -\omega \\ &\psi\big|_{\partial D} = 0 \\ &u(x,y) = -\frac{\partial \psi}{\partial y}, \quad v(x,y) = \frac{\partial \psi}{\partial x} \end{aligned} \tag{1}$$

where ω is the non-zero component of vorticity $\boldsymbol{\omega}(0,0,\omega)=\nabla\times\boldsymbol{u}$, ψ is the stream function. Equations (1) indicate that the vorticity field moves with the flow; so the evolution of vortex particles is expressed by equations:

$$\begin{aligned} \frac{dx_i}{dt} &= u_i = \left(\frac{\partial \psi}{\partial y}\right)_i \\ \frac{dy_i}{dt} &= v_i = -\left(\frac{\partial \psi}{\partial x}\right)_i \end{aligned} \tag{2}$$

The continued vorticity field is approximated by discreet distribution of the Dirac delta function (point vortices) :

$$\omega(x) = \sum_{i=1}^{N} \Gamma_i \delta(x - x_i), \quad \Gamma_i = \int_{A_i} \omega\, dxdy \tag{3}$$

where Γ_i is the intensity of a single vortex, A_i is the elementary area of the grid cell. In this way the evolution of vorticity is described by a system of N ordinary differential equations. In two dimensional space the vorticity field is replaced by a system of point vortices. This system of N-pointed vortices can be described by the Hamiltonian equations:

$$\Gamma_i \frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}$$
$$\Gamma_i \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i} \tag{4}$$

where $dH/dt = 0$. H is the Kirchoff-Routh function [8]:

$$H = -\frac{1}{2\pi} \sum_{i,j=1,i>j}^{N} \Gamma_i \Gamma_j \ln r_{ij} + \sum_{i=1}^{N} \gamma(x_i, y_i), \qquad r_{ij}^{\ 2} = (x_i - x_j)^2 + (y_i - y_j)^2 \tag{5}$$

The first term in the above formula is interpreted as a function of interaction between vortices, and we named it H_0; the other term is a harmonic function related to the presence of the solid boundary. When flow goes in unlimited 2D space then $\gamma \equiv 0$. In this paper the system of vortices "energetically" is characterized by H_0. Taking into account the fact that system (4) is Hamiltonian and describes the motion of a large number of the particles , it seems suitable to analyse the behaviour of vortex particles in terms of statistical physics. The purpose of this paper is to verify Onsager's hypothesis, which is connected with the "negative temperature" phenomenon.

3. "NEGATIVE TEMPERATURE" EFFECT

According to equation (4) the x and y coordinates of each vortex are canonical conjugates, so that the phase-space is identical with the configuration-space of the vortices [1]. It is assumed that the motion of each particle is ergodic. It means that all points of the surface of constant H in the phase-space are available with the same probability. If domain D is limited, then the phase-space is also limited. The volume of phase-space when E<H, entropy and temperature are defined:

$$\int_{-\infty}^{E} \Omega(E)dE = \int_{H<E} dx_1 dy_1 ... dx_N dy_N, \quad S = \log \Omega(E), \quad T = (dS/dE)^{-1} = \Omega/\Omega' \tag{6}$$

where $\Omega(E)$ is the state density function and the prime index mean differentiation.

If the system of vortices is energetically characterized by function H_0 [first term in formula (5)] then it is possible to obtain large values of energy H_0 by introduction of the order in the initial positions of the vortex particles. For example, particles of the same sign of intensity can be put close together. Onsager's hypothesis suggests that there exists the value E_m and when the system obtains $E > E_m$, then we have $dS/dE < 0$. So one can say that the system is in a "negative temperature" state. According to the hypothesis, the most likely state for a small value of energy H appears when vortices of the opposite sign are mixed. For large values of energy, when temperature is “negative”, vortices of the same sign tend to cluster. We tried to verify this hypothesis by numerical simulation.

4. ALGORITHM OF “VORTEX-IN-CELL” METHOD

The stream function was obtained by solving Poisson’s equation using a fast direct solver of the 4-th order (DFPS2H from IMSL library). All calculations were carried out in double precision. Calculations using the “vortex-in-cell” method were as follows [6,7]:

1) According to formula (3) in the time step $t^n = n\Delta t$ the intensity of each particle is redistributed onto nodes of the grid, and their vorticity $\omega(x_i)$ is calculated:

$$\Gamma_i = \sum_p \Gamma_p \, \varphi_i(x_p^n), \quad J_i = \sum_p h^2 \varphi_i(x_p^n), \quad \omega_i = \frac{\Gamma_i}{J_i} \tag{7}$$

where $\varphi_i(x) = \varphi((x\text{-}x_i)/h)$ is m-order B-spline [16]. For m = 1 ($\varphi_i(x) = 1 - |x|$ for $|x| \leq 1$, and for $|x| > 1$ $\varphi_i(x) = 0$); interpolation (7) corresponds to the area-weighted interpolation scheme. In the present paper for cells bordering the boundary of the domain, the first order B-spline were used. However, inside the domain the third order B-spline (m=3) was applied:

$$\varphi(x) = \begin{cases} \frac{1}{2}|x|^3 x^2 + \frac{2}{3}, & |x| \leq 1 \\ -\frac{1}{6}|x|^3 + x^2 - 2|x| + \frac{4}{3}, & 1 < |x| < 2 \\ 0, & |x| \geq 2 \end{cases} \tag{8}$$

It is important that the third order B-spline has twice as large as the first order one. This support for the interpolation function includes 16 nodes of the grid. B-splines are even functions and satisfy the normalisation condition: $\int \varphi(x) dx = 1$. The scheme described above is conservative and stable [6]:

$$\sum_j J_j \omega_j = \sum_p \Gamma_p, \qquad \sum_j J_j |\omega_j|^2 \leq \sum_p h^2 |\omega(x_p)|^2 \tag{9}$$

2) After redistribution of the vorticity, Poisson’s equation is solved:

$$\Delta\psi = -\omega, \qquad \psi|_{\partial D} = 0 \tag{10}$$

The velocity at the grid nodes is calculated by the central difference:

$$u_j = \frac{\Psi(x_{j_1}, y_{j_2} + \mathrm{h}) - \Psi(x_{j_1}, y_{j_2} - \mathrm{h})}{2\mathrm{h}}$$
$$v_j = -\frac{\Psi(x_{j_1} + \mathrm{h}, y_{j_2}) - \Psi(x_{j_1} - \mathrm{h}, y_{j_2})}{2\mathrm{h}} \qquad (11)$$

3) The velocity from the grid nodes is interpolated into the vortex particles:

$$u^n(x_p) = \sum_j u_j^n l_j(x_p) \qquad (12)$$

where $l_j(x)$ is the base function of the lagrangian interpolation. We used the second order interpolation implemented in the DQD2DR procedure of the IMSL library. A new vortex position in (n +1) time step is obtained when solving equations (2) by the fourth order Runge-Kutta method. It makes up one computational time step.

5. NUMERICAL RESULTS

We assumed that vortices were enclosed in square domain D 1x1. The time step was $\Delta t = 0.0005$ and the grid size was $\Delta x = \Delta y = 0.01$. At first we studied the evolution of the vortex point system for low energy. We took an approach similar to the work of Montgomery & Joyce [11, 12, 4]. We redistributed the vortex particles with the intensity $\Gamma_i = \pm 1/20000$ ($i = 1,2$) randomly over the domain. It gave the interaction energy $H_0 = -10^{-5}$. As we can see in Fig.1, distribution of the vortex point is nearly stationary. The particles moved along the stream lines, and the stream function after 5000 steps (Δt=0.0005) looks very similar as in t = 0.

Due to the fact that interaction energy H_0 is proportional to $\ln|x_i - x_j|$, and in order to obtain the higher value of H_0 , we gathered the vortex particles at sixteen square boxes with side a length equal to 19*dx (dx = 0.01). To the boxes we attached the constant vorticity $\omega_0 = \pm 10$. The sign of the vorticity attached to the boxes was changed alternately. Squares with the same sign of intensity are separated by squares the opposite sign. The global vorticity in the domain was zero. The vorticity inside the box was approximated by the random redistribution of 2500 point vortices. The number of vortex particles that took part in the computation was 16x25000=40000. The H_0 was equal to 4.330. The intensity of one vortex point was $|\Gamma_i| = 0.001$. By introducing order in the position of the particles, we decreased the entropy of the system. The low entropy and the high interaction energy caused the system to be far from equilibrium.

Fig.3A shows the sequence of the vortex particle distribution. We noticed that the vortex mergers and pairings, i.e. the formation of the dipolar structure, were made up of opposite-signed vortices. And one of the striking features of the picture sequence is the production of a thread–like structure that is called the filaments of vorticity. The filaments of vorticity are a characteristic feature of fully developed two-dimensional turbulent flow [5, 10]. The appearance of the filaments increases the tendency to reduce the interaction energy. To balance this tendency the vortex coherent structure must approach each other [1]. So one can say that filamentation is associated with vortex mergers [5]. It is worth noticing that the vorticity filaments are persistent and stable. Probably they are stabilized by large-scale vortex structures [5]. In the frame t = 15 of Fig.3A one can see that the vortex particles are segregated, thus being in agreement with Onsager's hypothesis. The final stage resembles the dipolar structure (the structure made up of opposite-sign coherent vortices which are separated by vorticity filaments (see also Fig.2, which shows compatibility of the vorticity distribution with the stream function in perspective 3D view). Fig.3B shows the exemplary distribution of stream function lines that correspond to vortex particle distribution at time t=0, t = 4 and t = 15. In that picture, the separation process of the vortex particles is clearly visible. We stopped the calculation because the monitored value of H_0 changed more then 25%. The variations of H_0 increased rapidly in the final steps of the calculation. We are aware that our numerical calculations introduced into the calculation some "numerical viscosity". The dissipation is greater when complex structures appears. We have to add that the transition process which we presented in Fig.3A is insensible in some intervals of variations of the numbers of particles and the size of the boxes. It depends only on the value of H_0 . Smaller values of ω_0 give smaller values of H_0, and causes the transition process to slow down.

6. CONCLUSIONS

The vortex method used to simulate the transition process in the distribution of the vorticity seems to be very attractive. The vortex particles are natural tools for the visualisation of the flow and dynamic of the vorticity fields. It seems that Onsager's hypothesis, that at a high energy state the vortices with the same sign will cluster, is correct. The merger process of the coherent vortex structure is accompanied by filamentation of the vorticity field.

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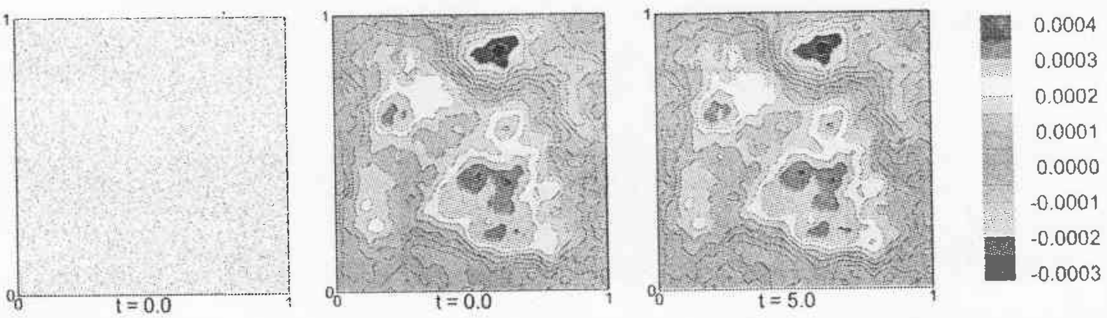

Figure 1: **Initial position of positive (red) and negative (green) vortices,** $H_0 = 10^{5}$**; initial stream function and nal stream function with contour legend.**

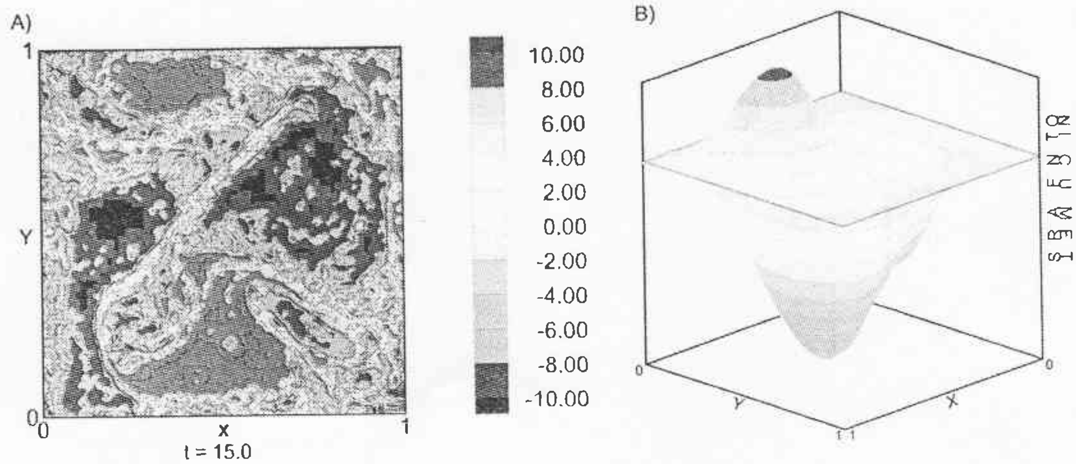

Figure 2: **A) Distribution of vorticity and its contour legend for** $H_0 = 4.330$**; B) Stream function related to the vorticity distribution from 2A.**

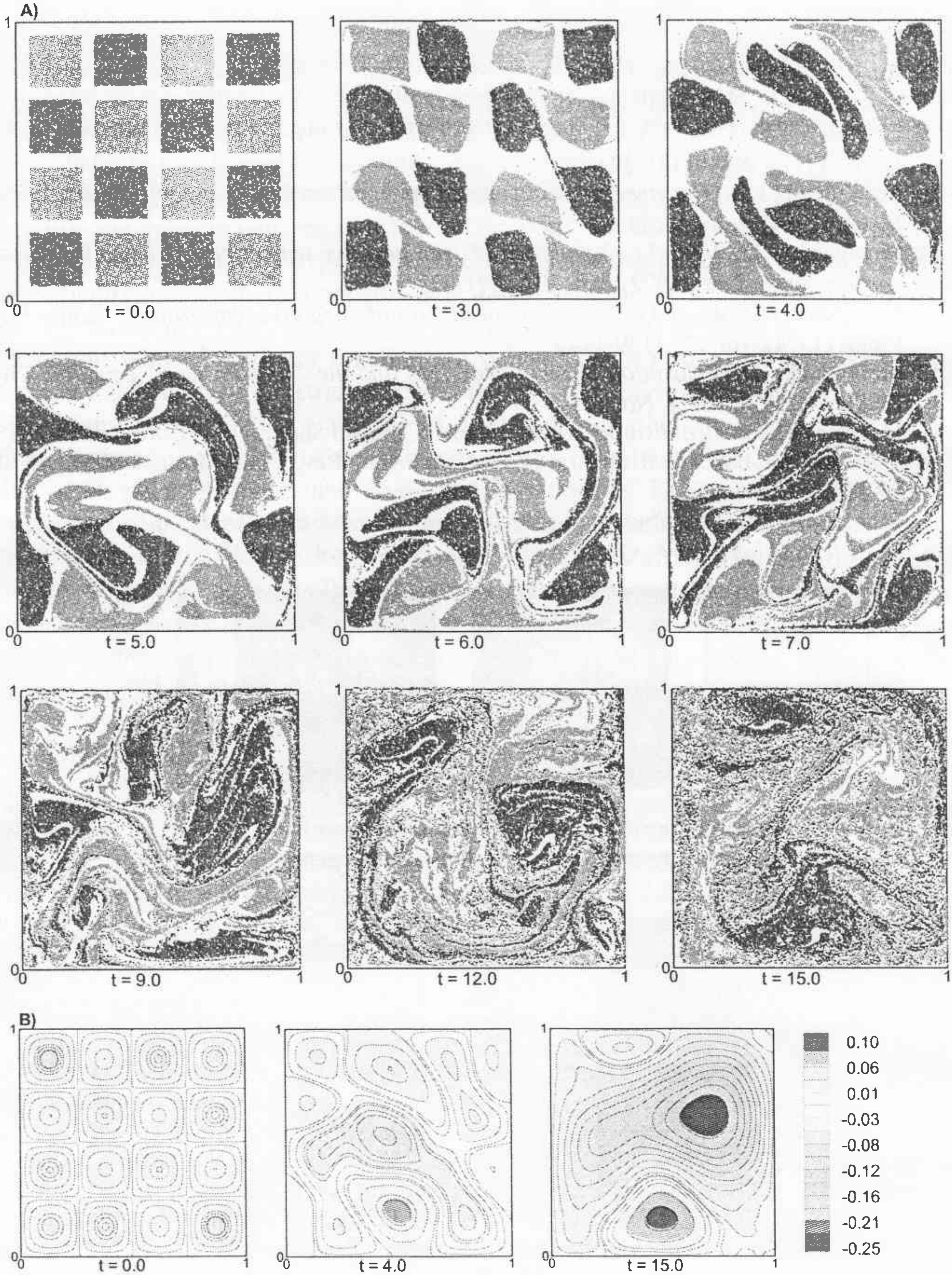

Figure 3: **A) Evolution of position of positive (red) and negative (green) vortex particles,** $H_0 = 4.330$; **B) Evolution of stream function related to vortices position for** $t = 0.0$; $t = 4.0$; $t = 15.0$.